# Transchromatic Generalized Characters

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# References I

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# Contents

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$$\chi : R(G) \to Cl(G; L)$$

This will induce an isomorphism

$$\chi : L \otimes R(G) \xrightarrow{\sim} Cl(G; L).$$

Moreover, the profinite integers $\hat{\mathbb{Z}}$ acts on $L$ and $G = \mathrm{Hom}(\hat{\mathbb{Z}}, G)$.

It acts on $f \in Cl(G; L)$ via

$$(\phi \circ f)(g) = \phi(f(\phi^{-1} g))$$

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$$(\phi \circ f)(g) = \phi(f(\phi^{-1}g))$$

The map $\chi$ actually lands in

$$\chi : R(G) \to Cl(G; L)^{\hat{\mathbb{Z}}},$$

which induces an isomorphism

$$\chi : \mathbb{Q} \otimes R(G) \xrightarrow{\sim} Cl(G; L)^{\hat{\mathbb{Z}}}.$$

Let $X$ be a $G$-space and $K_G^*(X)$ be set of (virtual) $G$-vector bundles over $X$.

We have $K_G^*(*) = R(G)$ and there is a natural map

$$R(G) \to K^*(BG)$$

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by applying the Borel construction $EG \times_G -$.

The completion theorem tells us

$$R(G)_I^\wedge \xrightarrow{\simeq} K^*(BG).$$

In general, the projection map

$$\pi : EG \times X \to X$$

will induces an isomorphism

$$K_G^*(X)_I^\wedge \xrightarrow{\sim} K_G^*(EG \times X) = K^*\left((EG \times X)/G\right).$$

Recall that $K$ is of height 1. Can we mimic these behaviors in a theory of height $n$ with $K*(EG \times_G -)$ replaced by $E^*(EG \times_G -)$?

Recall that $K$ is of height 1. Can we mimic these behaviors in a theory of height $n$ with $K*(EG \times_G -)$ replaced by $E^*(EG \times_G -)$?

- $E$-complex oriented, with $E^*$ complete local w.r.t $\mathfrak{m}$.
- The residue field $E^*/\mathfrak{m}$ has char$= p > 0$ and $p^{-1}E^* \neq 0$.
- The formal group $\mathbb{G}_E$ has height $n$ over the residue field.

# $L(E^*)$–The analogue of $L$

The inverse system

$$\cdots \to (\mathbb{Z}/p^{r+1})^n \to (\mathbb{Z}/p^r)^n \to \cdots$$

induces a direct system

$$\cdots \to E^*(B(\mathbb{Z}/p^r)^n) \to E^*(B(\mathbb{Z}/p^{r+1})^n) \to \cdots$$

We let $E^*_{cont}(B\mathbb{Z}_p^n)$ denote this colimit.

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For any nonzero $\alpha : (\mathbb{Z}/p^r)^n \to S^1$, we can form an element

$$c_1(\alpha) = \alpha^*(x), \alpha^* : E^*(BS^1) \to E^*(B\Lambda_r).$$

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$$c_1(\alpha) = \alpha^*(x), \alpha^* : E^*(BS^1) \to E^*(B\Lambda_r).$$

Let $L_r(E^*) = S^{-1}E^*(B\Lambda_r)$, with $S$ generated by such $c_1(\alpha)$.

# $L(E^*)$–The analogue of $L$

The ring $L_r(E^*)$ is flat over $E^*$. If $\alpha$ is a generator of $(\mathbb{Z}/p^r)^*$, and we write $x$ for $c_1(\alpha)$, then

$$E^*(B\mathbb{Z}/p^r) = E^*[\![x]\!]/[p^r](x).$$

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In particular $p$ is invertible in $L_r(E^*)$.
Note that this ring $L_r(E^*)$ can be acted by $\mathrm{Aut}(\Lambda_r)$, and the map

$$L_r(E^*) \to L_{r+1}(E^*)$$

is an $\mathrm{Aut}(\Lambda_{r+1})$ equivariant map, via the projection $\mathrm{Aut}(\Lambda_{r+1}) \to \mathrm{Aut}(\Lambda_r)$ acting on domain.

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is an $\mathrm{Aut}(\Lambda_{r+1})$ equivariant map, via the projection $\mathrm{Aut}(\Lambda_{r+1}) \to \mathrm{Aut}(\Lambda_r)$ acting on domain.
Taking direct limit we have $L(E^*)$ is acted by $\mathrm{Aut}(\mathbb{Z}_p^n)$, and

$$L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^n)} = p^{-1}E^*.$$

# $L(E^*)$–The analogue of $L$

There is an 1-1 correspondence from $\mathrm{Hom}(\Lambda_r^*, \mathbb{G}_E[p^r])(R)$ to the set of maps

$$\theta : R[\![x]\!]/[p^r](x) \to R^{\Lambda_r^*}.$$

To be explicit, let $\phi : \Lambda_r^* \to \mathrm{Hom}(E^*[\![x]\!]/[p^r](x), R)$. Then

$$\theta : x \mapsto (\phi(a_1)(x), \phi(a_2)(x), \cdots)$$

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$$\theta : x \mapsto (\phi(a_1)(x), \phi(a_2)(x), \cdots)$$

**Proposition**

*$\theta$ is an isomorphism $\Longleftrightarrow$ all $\phi(a_i)(x)$ are units.*

# $L(E^*)$–The analogue of $L$

The ring $L_r(E^*)$ and $L(E^*)$ have interesting moduli interpretations.

$E^*(B\Lambda_r)$ corepresents the functor $\mathrm{Hom}(\Lambda_r^*, \mathbb{G}_E[p^r])$

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$L(E^*)$ is the initial extension of $E^*$ such that the base change of $\mathbb{G}_E[p^\infty]$ becomes trivial, i.e. $(\mathbb{Q}_p/\mathbb{Z}_p)^n$.

# $Cl_{n,p}(G,X;L(E^*))$–The analogue of $Cl(G;L)$

For each $\alpha \in \mathrm{Hom}(\Lambda_r, G)$, we have an induced morphism

$$B\Lambda_r \times X^{Im(\alpha)} \to EG \times_G X.$$

$$E^*(EG \times_G X) \to E^*(B\Lambda_r \times X^{Im(\alpha)}) = L_r(E^*) \otimes_{E^*} E^*(X^{Im(\alpha)}).$$

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For $r$ sufficient large, $\mathrm{Hom}(\Lambda_r, G) = \mathrm{Hom}(\mathbb{Z}_p^n, G) = G_{n,p}$.

$$\mathrm{Fix}_{n,p}(G, X) := \coprod_{\alpha \in G_{n,p}} X^{Im(\alpha)}$$

Hence we obtain a map

$$\chi_{n,p}^G : E^*(EG \times_G X) \to L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X)).$$

## $Cl_{n,p}(G,X;L(E^*))$–The analogue of $Cl(G;L)$

The group $G \times \Lambda_r$ acts on $L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X))$.

# $Cl_{n,p}(G,X;L(E^*))$–The analogue of $Cl(G;L)$

The group $G \times \Lambda_r$ acts on $L_r(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X))$.

**Proposition**

*$\chi^G_{n,p}$ actually lands in $G \times \Lambda_r$ invariants.*

**Proof.**

Following diagrams commute, $\alpha \in G_{n,p}$ and $\phi \in \mathrm{Aut}(\Lambda_r)$.

$$\begin{array}{ccc}
B\Lambda_r \times X^{Im(\alpha\circ\phi)} & \xrightarrow{\phi\circ 1} & B\Lambda_r \times X^{Im(\alpha)} \\
\downarrow{\scriptstyle \alpha\circ\phi} & & \downarrow{\scriptstyle \alpha} \\
EG \times_G X & \xlongequal{\quad} & EG \times_G X
\end{array}$$

□

# $Cl_{n,p}(G,X;L(E^*))$–The analogue of $Cl(G;L)$

Thus we obtain the desired generalized character map

$$\chi^G_{n,p} : E^*(EG \times_G X) \to Cl_{n,p}(G,X;L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^n)}$$

where

$$Cl_{n,p}(G,X;L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G,X))^G.$$

# $Cl_{n,p}(G, X; L(E^*))$–The analogue of $Cl(G; L)$

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where

$$Cl_{n,p}(G, X; L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, X))^G.$$

Recall that $L(E^*)$ is finite faithfully flat over $p^{-1}E^*$, hence it defines a cohomology theory of height 0.

## Theorem (Hopkins, Kuhn, Ravenel)

*The generalized character map $\chi_{n,p}^G$ induces isomorphisms*

$$\chi_{n,p}^G : L(E^*) \otimes_{E^*} E^*(EG \times_G X) \to Cl_{n,p}(G, X; L(E^*)),$$

*and*

$$\chi_{n,p}^G : p^{-1}E^* \otimes_{E^*} E^*(EG \times_G X) \to Cl_{n,p}(G, X; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^n)}.$$

**Theorem (Hopkins, Kuhn, Ravenel)**

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$$\chi^G_{n,p} : p^{-1}E^* \otimes_{E^*} E^*(EG \times_G X) \to Cl_{n,p}(G, X; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^n)}.$$

When $X$ is a point, $\mathrm{Fix}_{n,p}(G, *) = G_{n,p}$, hence

$$Cl_{n,p}(G, *; L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{n,p}(G, *))^G,$$

which is the orbit of $|G_{n,p}|$ copies of $L(E^*)$ under the action of $G$. It can also be identified with the ring of functions from $G_{n,p}$ to $L(E^*)$ stable under $G-$orbits.

# Part of the proof

Consider both side as cohomology theory from the category of pairs $(G, X)$.

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Consider both side as cohomology theory from the category of pairs $(G, X)$.

Under some technical conditions, reduce to show the case $G$ is abelian and $X$ is a point.

$$\chi_{n,p}^{A} : L(E^*) \otimes_{E^*} E^*(BA) \to L(E^*)^{|\mathrm{Hom}(\Lambda, A)|}$$

Can we do the similar things, starting from a height $n$ theory, namely $E_n$, but end up with a height $n - t > 0$ theory?

Can we do the similar things, starting from a height $n$ theory, namely $E_n$, but end up with a height $n-t>0$ theory?

The ring $L(E^*)$ should be replaced by an $L_t = L_{K(t)}E_n^0$ algebra, such that the base change of the $p$-divisible group $\mathbb{G}_{E_n}[p^\infty]$ becomes constant (partly).

# The algebra $C_t$

Suppose $\mathbb{G}_{E_n}$ be the formal/$p$-divisible group over $E_n^*$ and $\mathbb{G} = L_t \otimes \mathbb{G}$. Let $\mathbb{G}_0$ be the formal/$p$-divisible group $\mathbb{G}_{L_{K(t)} E_n}$.

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By Weierstrass preparation, we have

$$[p^k]_{\mathbb{G}_{E_n}}(x) = f_k(x) \cdot unit \in E^0[\![x]\!]$$
$$[p^k]_{\mathbb{G}_{L_{K(t)}E_n}}(x) = g_k(x) \cdot unit \in L_t[\![x]\!]$$

with $\deg f_k = p^{kn}$ and $\deg g_k = p^{kt}$.

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with $\deg f_k = p^{kn}$ and $\deg g_k = p^{kt}$.

It follows that $g_k$ divides $f_k$, hence $\mathbb{G}_0$ is a sub $p$-divisible group of $\mathbb{G}$.

# The algebra $C_t$

In fact, we have an exact sequence

$$0 \to \mathbb{G}_0 \to \mathbb{G} \to \mathbb{G}_{\acute{e}t} \to 0$$

between $p$-divisible groups over $\mathrm{Spf}(L_t)$, with height $\mathbb{G}_{\acute{e}t} = n - t$.

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between $p$-divisible groups over $\mathrm{Spf}(L_t)$, with height $\mathbb{G}_{\acute{e}t} = n - t$.

Let $\Lambda_r = (\mathbb{Z}/p^r)^{n-t}$, recall that over $E^0$ algebras,

$$\mathrm{Hom}_{E^0}(E^0(B\Lambda_r), -) = \mathrm{Hom}(\Lambda_r^*, \mathbb{G}_{E_n}[p^r]).$$

Hence over $L_t$ algebras, we have

$$\mathrm{Hom}_{L_t}(L_t \otimes_{E^0} E^0(B\Lambda_r), -) = \mathrm{Hom}(\Lambda_r^*, \mathbb{G}[p^r]).$$

# The algebra $C_t$

Let $C_r'$ denote the ring $L_t \otimes_{E^0} E^0(B\Lambda_r)$. Over $C_r'$ we have a canonical morphism

$$\mathbb{G}_0[p^r] \oplus (\mathbb{Z}/p^r)^{n-t} \to \mathbb{G}[p^r],$$

# The algebra $C_t$

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which induces a canonical morphism

$$\phi : \Lambda_r^* = (\mathbb{Z}/p^r)^{n-t} \to \mathbb{G}_{\acute{e}t}[p^r].$$

# The algebra $C_t$

Let $C'_r$ denote the ring $L_t \otimes_{E^0} E^0(B\Lambda_r)$. Over $C'_r$ we have a canonical morphism

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which induces a canonical morphism

$$\phi : \Lambda_r^* = (\mathbb{Z}/p^r)^{n-t} \to \mathbb{G}_{\acute{e}t}[p^r].$$

Let $S$ be the multiplicative closed subset generated by $\phi(\Lambda_r^*) \subset \mathbb{G}_{\acute{e}t}[p^r](C'_r)$ and denote $S^{-1}C'_r$ by $C_r$.

# The algebra $C_t$

The $L_t$ algebra $C_r$ is the initial object which carries an isomorphism

$$\mathbb{G}_0[p^r] \oplus (\mathbb{Z}/p^r)^{n-t} \xrightarrow{\sim} \mathbb{G}[p^r],$$

and $C_t = \operatorname{colim}_r C_r$ is the initial object which carries an isomorphism

$$\mathbb{G}_0 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t} \xrightarrow{\sim} \mathbb{G}.$$

Let $X$ be a $G$-space.

$$\mathrm{Fix}_{n-t}(G,X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^{n-t},G)} X^{Im(\alpha)}.$$

Each $\alpha$ induces $B\Lambda_r \times X^{Im(\alpha)} \to EG \times_G X$, and hence

$$B\Lambda_r \times \mathrm{Fix}_{n-t}(G,X) \to EG \times_G X,$$

$$\begin{aligned} E_n^*(EG \times_G X) &\to E_n^*(B\Lambda_r) \otimes_{E_n^*} E_n^*(EG \times_G \mathrm{Fix}_{n-t}(G,X)) \\ &\to C_r^* \otimes_{L_t^*} L_{K(t)} E_n^*(EG \times_G \mathrm{Fix}_{n-t}(G,X)). \end{aligned}$$

Let $C_t^*(X)$ denote $C_t^* \otimes_{L_t^*} L_{K(t)} E_n^*(X)$

**Theorem (Stapleton)**

*There is a character map* $\Phi_t^G$

$$\Phi_t^G : E_n^*(EG \times_G X) \to C_t^*(EG \times_G \mathrm{Fix}_{n-t}(G, X))$$

*which induces an isomorphism*

$$\Phi_t^G : C_t \otimes_{E_n^0} E_n^*(EG \times_G X) \to C_t^*(EG \times_G \mathrm{Fix}_{n-t}(G, X))$$

# Thank You!