Persistent Homology in Time Series Analysis and its Application to Wheeze Detection

Presenter: Yifan Wu

Advisor: Yifei Zhu

Content

References

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Dynamical Systems

- A ***global continuous time dynamical system*** is a pair (M, Φ), where M is a topological space and $\Phi : \mathbb{R} \times M \longrightarrow M$ is a continuous map so that $\Phi(0, p) = p$, and $\Phi\big(s, \Phi(t, p)\big) = \Phi(s + t, p)$ for all $p \in M$ and all $t, s \in \mathbb{R}$.
- $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and $\Phi\big(t, (a, b)\big) = (a + t, b + \alpha t)$. If α is rational, then every orbit is periodic. Otherwise every orbit is dense in $\mathbb{T}^2$.
- A solution to an differential equation is a dynamical system, for instance:

Dynamical Systems

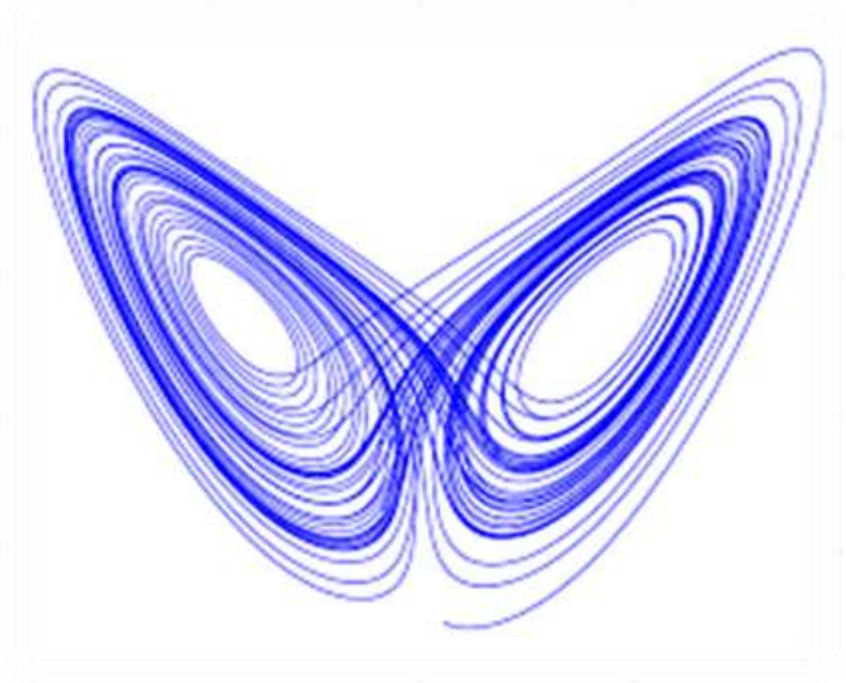
$$x'(t) = \sigma \cdot (y - x)$$
$$y'(t) = x \cdot (\rho - z) - y$$
$$z'(t) = xy - \beta z$$

- A subset $A \subset M$ called ***attractor*** is especially important since it attracts the evolution of states in close proximity.
- A is compact.
- A is an ***invariant*** set, i.e. $\Phi(t, A) \subset A$ for $t \geq 0$.
- There is an invariant open neighborhood U (called ***the basin of attraction***) of A, so that:

$$\cap_{t \geq 0} \Phi(t, U) = A.$$

Dynamical Systems

- $\mathbb{T}^2$ is the attractor of itself if α is irrational in the previous example.
- Lorenz's butterfly attractor.



Dynamical Systems

- The shape of an attractor is crucial.
  - Circle implies periodic process.
  - Non-integral Hausdorff dimension implies chaos.
  - High-dimensional tori $\mathbb{T}^n$ implies quasiperiodicity.

- It is difficult to achieve this goal for some reasons.
  - There is no precise description for the state space M.
  - How to figure out the “shape”?

Taken's Theorem

- Weather may be regarded as a dynamical system, but one can not give a precise description about it. Instead, one can easily obtain measurements of relevant quantities for each state $p \in M$, for instance temperature, pressure, etc.
- A way of measuring can be thought of as a continuous map $F: M \longrightarrow \mathbb{R}$ called an ***observation function***. For a given initial state $p \in M$, one obtains the scalar time series

$$\varphi_p : \mathbb{R} \longrightarrow \mathbb{R},$$

$$t \mapsto F \circ \Phi(t, p).$$

Taken’s Theorem

- Let M be a smooth, compact, Riemann manifold; let $\tau > 0$ be a real number; and let $d \geq 2\text{dim}(M)$ be an integer. Then for generic $\Phi \in C^2(\mathbb{R} \times M, M)$ and $F \in C^2(M, \mathbb{R})$, and for $\varphi_p(t)$ defined above, the delay map $\psi : M \longrightarrow \mathbb{R}^{d+1}$

$$p \mapsto (\varphi_p(0), \varphi_p(\tau), \varphi_p(2\tau), \dots, \varphi_p(d\tau))$$

is an embedding.

- Generic means that Φ, F are both open and dense in Whitney topology.

Sliding Window Embedding

- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, $\tau > 0$ a real number, and $d > 0$ an integer. The ***sliding window embedding*** of f, with paremeters d and τ, is the vector valued function

$$SW_{d,\tau}f : \mathbb{R} \longrightarrow \mathbb{R}^{d+1},$$

$$t \mapsto \big(f(t), f(t+\tau), f(t+2\tau), \ldots, f(t+d\tau)\big).$$

- For $T \subset \mathbb{R}$, the set $\mathbb{SW}_{d,\tau}f = \left\{SW_{d,\tau}f(t) : t \in T\right\}$ is the ***sliding window point cloud*** associated to the ***sampling set*** T.

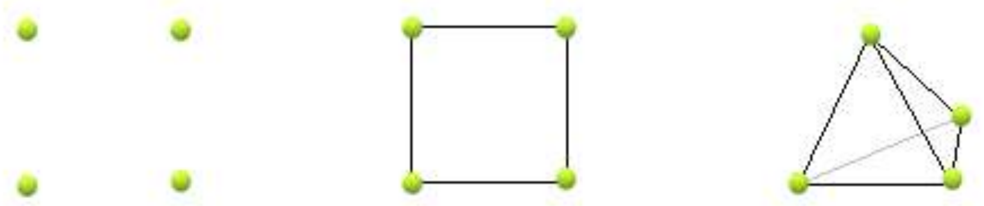
Sliding Window Embedding

- Thus, given time series data $f(t) = \varphi_p(t)$ observed from a potentially unknown dynamical system (M, Φ), Taken's theorem implies that the sliding window point cloud $\mathbb{SW}_{d,\tau}f$ provides a topological copy of $\{\Phi(t, p): t \in T\}$.
- Therefore, one successfully carries the orbit of a point $p \in M$ homeomorphically to an Euclidean space!

Rips Complex

- For a finite set of points $\mathcal{X} = \{x_i\}$, we can form a simplicial complex (abstract) called ***Rips complex*** $\mathcal{R}_\alpha(\mathcal{X})$:

$$\mathcal{R}_\alpha(\mathcal{X}) = \left\{ \left\{ x_{i_1}, \cdots, x_{i_k} \right\} \subset \mathcal{X} : d\left(x_{i_p}, x_{i_q} \right) \leq \alpha, 0 \leq p, q \leq k \right\}.$$



Rips Complex

- Let $H_n(\mathcal{R}_\alpha(\mathcal{X});\mathbb{F})$ denote the n'th homology group (vector space) of Rips complex of $\mathcal{X}$ with coefficients in filed $\mathbb{F}$, usually $\mathbb{F}_2$. Let $\beta_n(\mathcal{R}_\alpha(\mathcal{X});\mathbb{F})$ denote the rank (dimension) of $H_n(\mathcal{R}_\alpha(\mathcal{X});\mathbb{F})$, called the ***n'th Betti number*** of $\mathcal{R}_\alpha(\mathcal{X})$. Betti number implies the number of n-dimensional holes in a given topological space.
- If $\alpha \leq \alpha'$, there is an inclusion $\iota^{\alpha,\alpha'}$ from $\mathcal{R}_\alpha(\mathcal{X})$ to $\mathcal{R}_{\alpha'}(\mathcal{X})$,

$$\iota^{\alpha,\alpha'}\left(\left\{x_{i_1},\cdots x_{i_k}\right\}\right) = \left\{x_{i_1},\cdots x_{i_k}\right\},$$

and induces a linear transformation:

$$\iota_n^{\alpha,\alpha'}: H_n(\mathcal{R}_\alpha(\mathcal{X});\mathbb{F}) \longrightarrow H_n(\mathcal{R}_{\alpha'}(\mathcal{X});\mathbb{F}).$$

Persistent Homology

- A ***persistence vector space*** $\mathbb{V}$ is a collection of vector spaces V_α, $\alpha \in \mathbb{R}$, and linear transformations $\iota^{\alpha,\alpha'}: V_\alpha \longrightarrow V_{\alpha'}, \alpha \leq \alpha'$, so that:

1. $\iota^{\alpha,\alpha}$ is the identity map of V_α for all α.

2. $\iota^{\alpha',\alpha''} \circ \iota^{\alpha,\alpha'} = \iota^{\alpha,\alpha''}$ for all $\alpha \leq \alpha' \leq \alpha''$.

- If $\gamma \in V_\alpha$ is a nonzero element, then define

$$birth(\gamma) = \inf \{\tilde{\alpha} \leq \alpha: \gamma \in \mathrm{Im}(\iota^{\tilde{\alpha},\alpha})\}$$

$$death(\gamma) = \sup \{\alpha' \geq \alpha: \gamma \notin \mathrm{Ker}(\iota^{\alpha,\alpha'})\}$$

$$persistence(\gamma) = death(\gamma) - birth(\gamma)$$

Persistent Homology

- We will use $bcd_n^{\mathcal{R}}(\mathcal{X};\mathbb{F})$ to denote the barcode for the n-dimensional persistent homology of $\mathcal{R}(\mathcal{X})$.

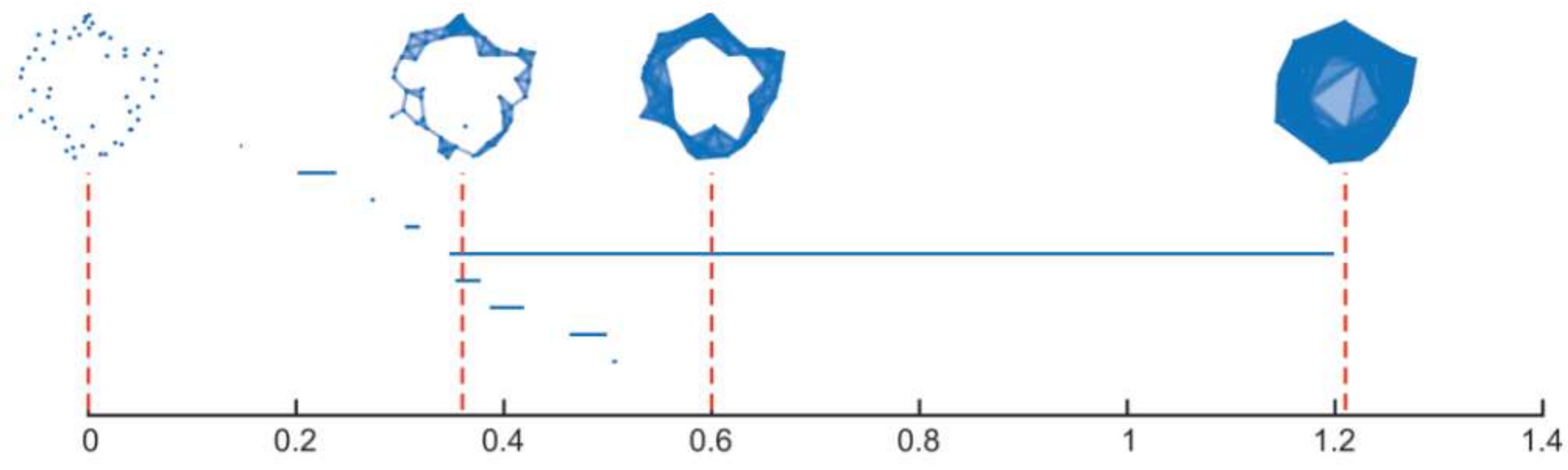


Figure 1.[Perea, 2019] Barcode for the Rips filtration of $\mathcal{X} \subset \mathbb{R}^2$ near S^1.

Persistent Homology

- Let $\mathbb{V}$ be a persistence vector space so that $\dim(V_\alpha)$ is finite for all α. Then there exists a multiset of intervals I called the barcode of $\mathbb{V}$, denoted $bcd(\mathbb{V})$, and so that:

1. For all α, $\dim(V_\alpha)$ is exactly the number of intervals $I \in bcd(\mathbb{V})$, counted with repetitions, with $\alpha \in I$.
2. For every $I \in bcd(\mathbb{V})$, and $\alpha \in I$, there is $\gamma \in V_\alpha$ with the left and right end-points of I are $birth(\gamma)$ and $death(\gamma)$.

- $H_n(\mathcal{R}_\alpha(\mathcal{X}); \mathbb{F})$ together with maps $\iota_n^{\alpha,\alpha'}$ is a persistence vector space, called the ***n-dimensional persistent homology***, with coefficients in $\mathbb{F}$, of the ***Rips filtration*** $\mathcal{R}(\mathcal{X}) = \{\mathcal{R}_\alpha(\mathcal{X})\}_{\alpha \in \mathbb{R}}$.

Persistent Homology

- Let $\omega = \sqrt{3}$ be irrational. Consider the dynamics Φ and observation function F on torus $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{C}^2$, given by

$$\Phi : \ (t, (z_1, z_2)) \longrightarrow (e^{it} z_1, e^{i\omega t} z_2)$$

$$F : (z_1, z_2) \longrightarrow \mathrm{Re}(z_1 + z_2).$$

Let $p = (1,1)$, $d = 4$, $\tau = \frac{3}{4}\sqrt{3}\pi$, $n = 0,1,2$.

Persistent Homology

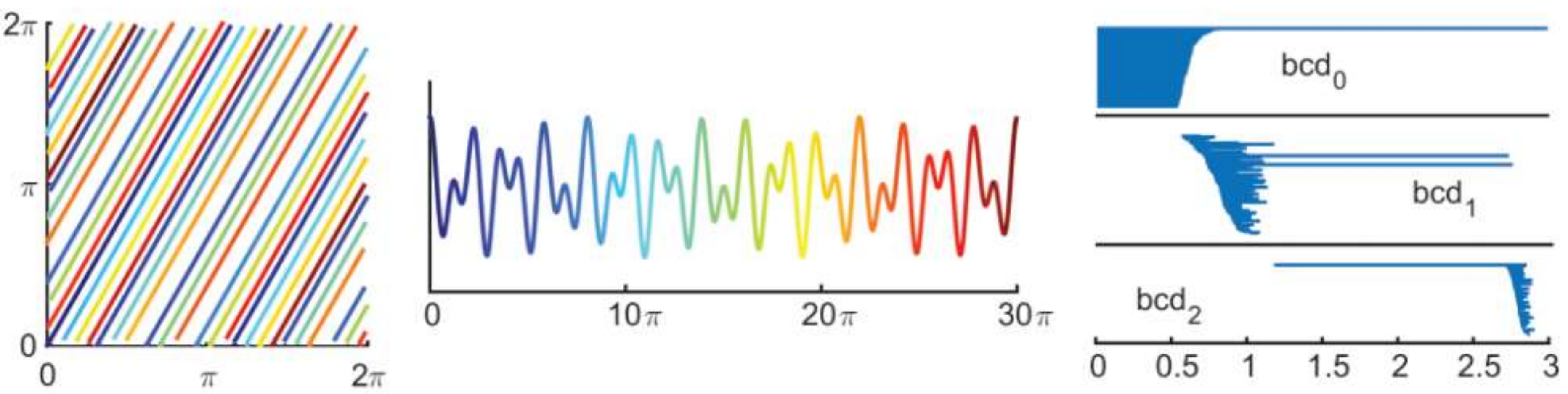


Figure 2.[Perea, 2019]
Left: The orbit of point p. The colors, blue through red, indicate the time.
Center: The time series $f(t) = F \circ \Phi(t, p) = \cos t + \cos \sqrt{3}t$.
Right: Barcodes for the Rips filtration $\mathcal{R}(\mathbb{SW}_{d,\tau} f)$, the number of long intervals recovers the Betti numbers of the orbit: $\beta_0 = \beta_2 = 1$; $\beta_1 = 2$.

Application to Wheeze Detection

- For a scalar time series $\{x_i\}$, a representation of the delay coordinate embedding can be described as the following vector quantity of *m* components:

$$X_i = (x_i, x_{i+j}, x_{i+2j}, \ldots, x_{i+(m-1)j})$$

  Where *j* is the index delay and *m* is the embedding dimension. If the sampling time is T_s, then the delay time τ is connected to the index delay *j* by the equality $\tau = j \cdot T_s$. We will use $m = 2$ in the rest of our discussion.

- Discrete time sound signals can be considered as a series expressing the amplitude of the wave in volts at each time instance, i.e. $x(t_i) = x_i$, *i=1,2,...k* and $t_i = i \cdot T_s$.

Application to Wheeze Detection

- Taken’s theorem implies that this embedding carries the whole topological data.
- Thus for a given discrete time sound signal $x(t_i)$, we can embed it into $\mathbb{R}^2$, then use the proposed persistent homology method to suggest the shape of the original signal.
- There are still something to be done, namely, selecting time delay and wheeze modeling.

Selecting Time Delay

- Examine an autocorrelation-like function(ACL) to choose a proper delay. The ACL function of a non-stationary digital signal $x(t_i)$ is calculated as follows:

$$R_{xx}(t_i) = \sum_{1 \le l \le k} x(t_i) \cdot x(t_l)$$

- According to experimental results, the appropriate interval for choosing delay time is $t_{c1} < \tau < t_{c2}$, where t_{c1} and t_{c2} are the first and second critical points of the ACL function $R_{xx}(t_i)$.

Selecting Time Delay

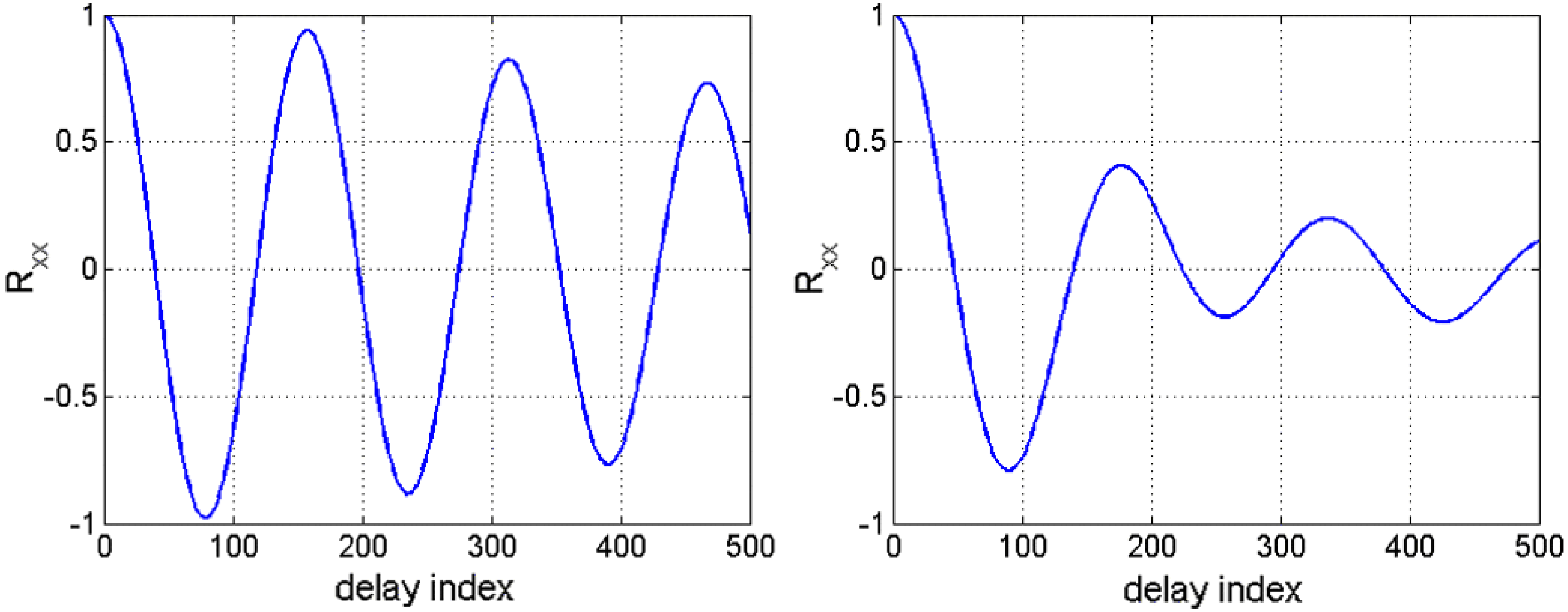


Figure 3.[Emrani Saba, 2014] The ACL function of a non-wheeze signal (left) and a wheeze signal (right).

3.2. Wheeze Modeling

- We propose a model for wheeze signals in time domain defined as a continuous piecewise sinusodal function with different periods and phases and a time varying amplitude, represented as

$$\omega(t) = \sum_{i=1}^{n} g_i(t)$$

$$g_i(t) = \begin{cases} A(t)\sin\left(\frac{2\pi}{T_i}t + \phi_i\right), t_{i-1} \le t \le t_i \\ 0 \end{cases}$$

- For each discrete wheeze signal denote by $s(t)$, we can construct such a $\omega(t)$.

3.2. Wheeze Modeling

- This model performs well on wheeze signals but not good on non-wheeze signals.

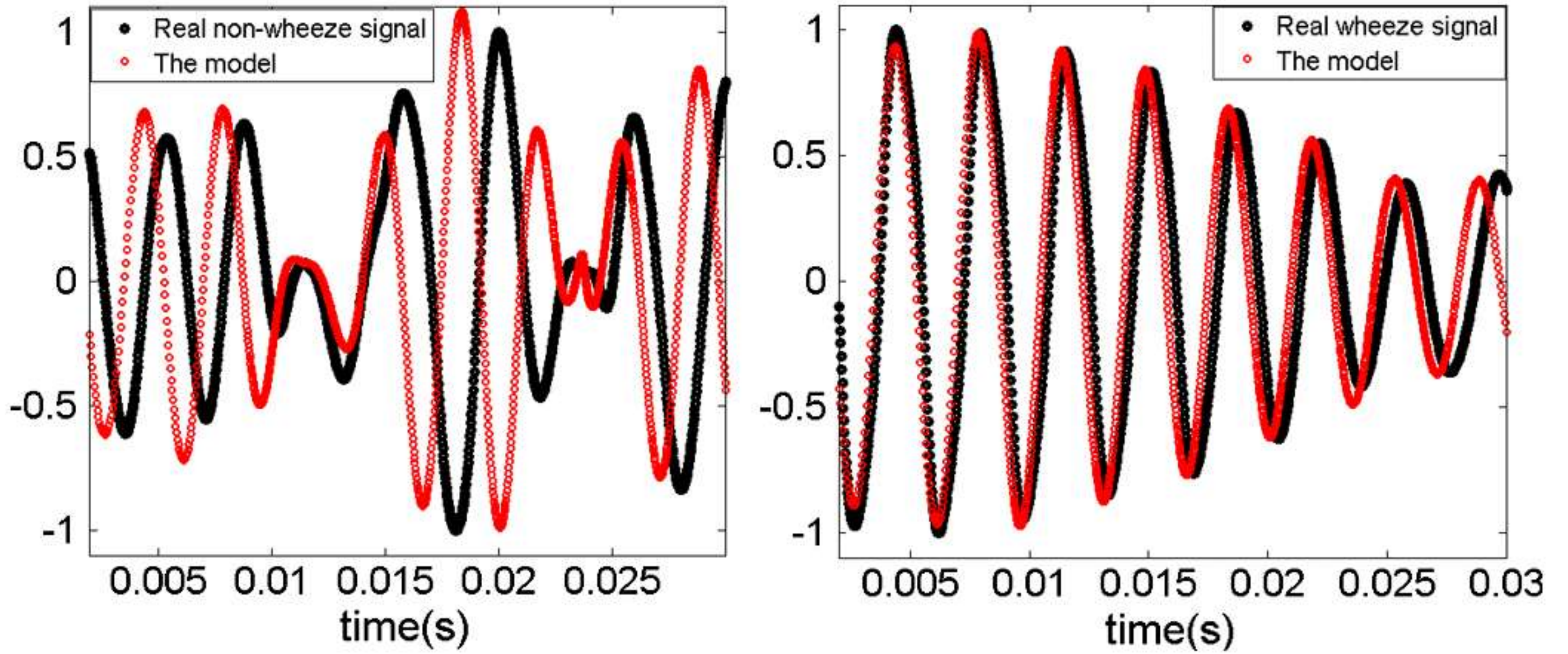


Figure 4.[Emrani Saba, 2014] Comparison of the proposed model for a non-wheeze breathing sound signal (left) and a wheeze signal (right).

3.2. Wheeze Modeling

- Now we turn to embed $\omega(t)$ and apply our method. To be precise, the continuous time delay embedding of $\omega(t)$ and $\omega_i(t)$ can be defined as:

$$W(t) = \{(\omega(t), \omega(t+\tau)) : t \in \mathbb{R}\}$$

$$W_i(t) = \{(\omega_i(t), \omega_i(t+\tau)) : t_{i-1} \le t \le t_i\}$$

- Lemma : The time delay embedding of a sinusoidal function with a suitable time delay is an ellipse.

3.2. Wheeze Modeling

- The time delay embedding W of a wheeze signal is similar to the union of W_i with few point missed. And therefore, the Rips complex associated to the corresponding point cloud of W is close to a set of concentric ellipse with angles of rotation $\pm 45°$, which always has at least one 1-dimensional persistent hole.
- On the other hand, using experimental results, the one shows that the first persistent Betti number of the delay embedding of a non-wheeze signal is zero.

3.3. Experimental Results

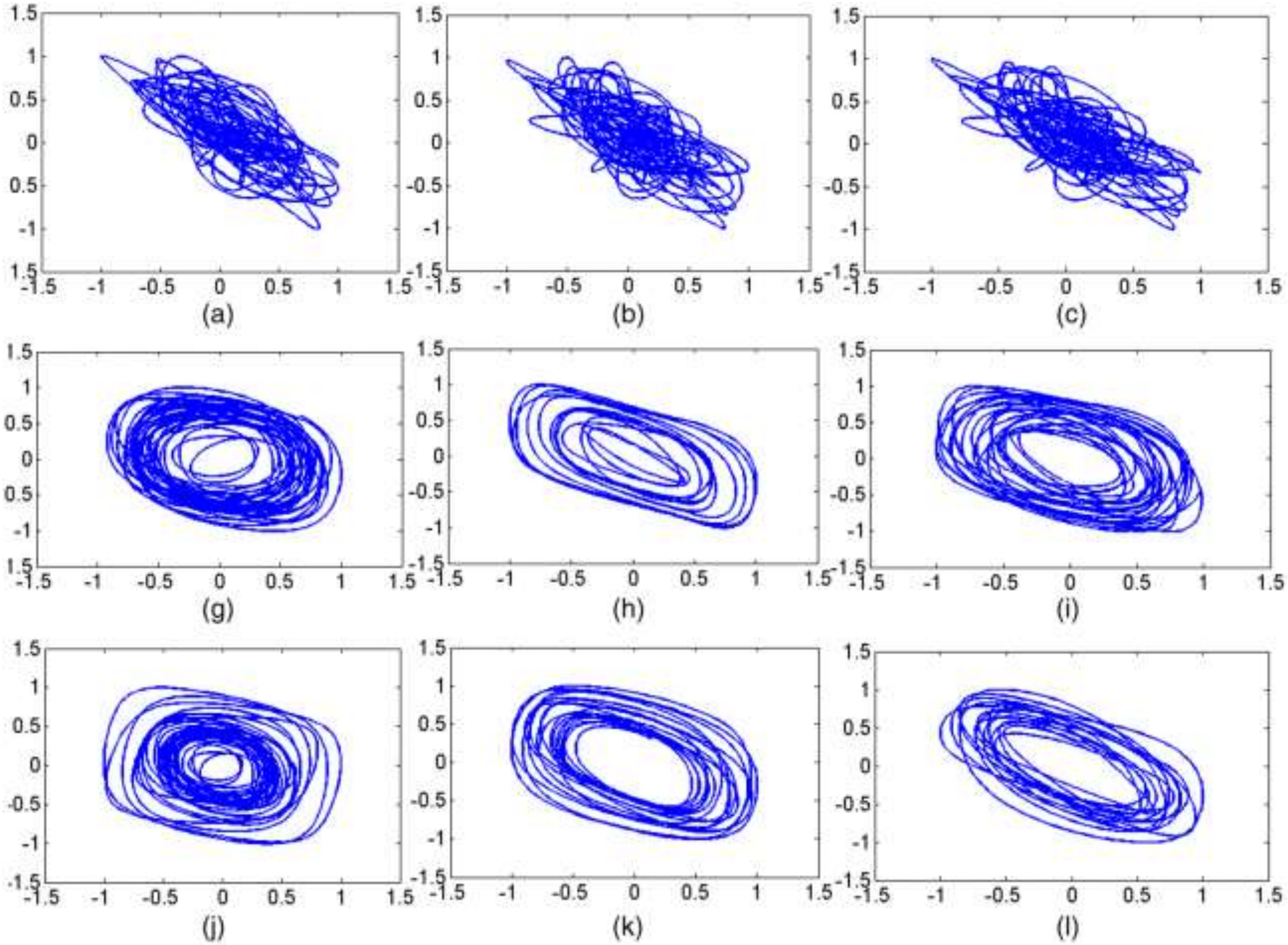


Figure 5.[Emrani Saba, 2014] Delay-embedding of non-wheeze signals recorded over (a) apex, (b) midlung and (c) chest. Delay embeddings of some wheeze signals.

3.3. Experimental Results

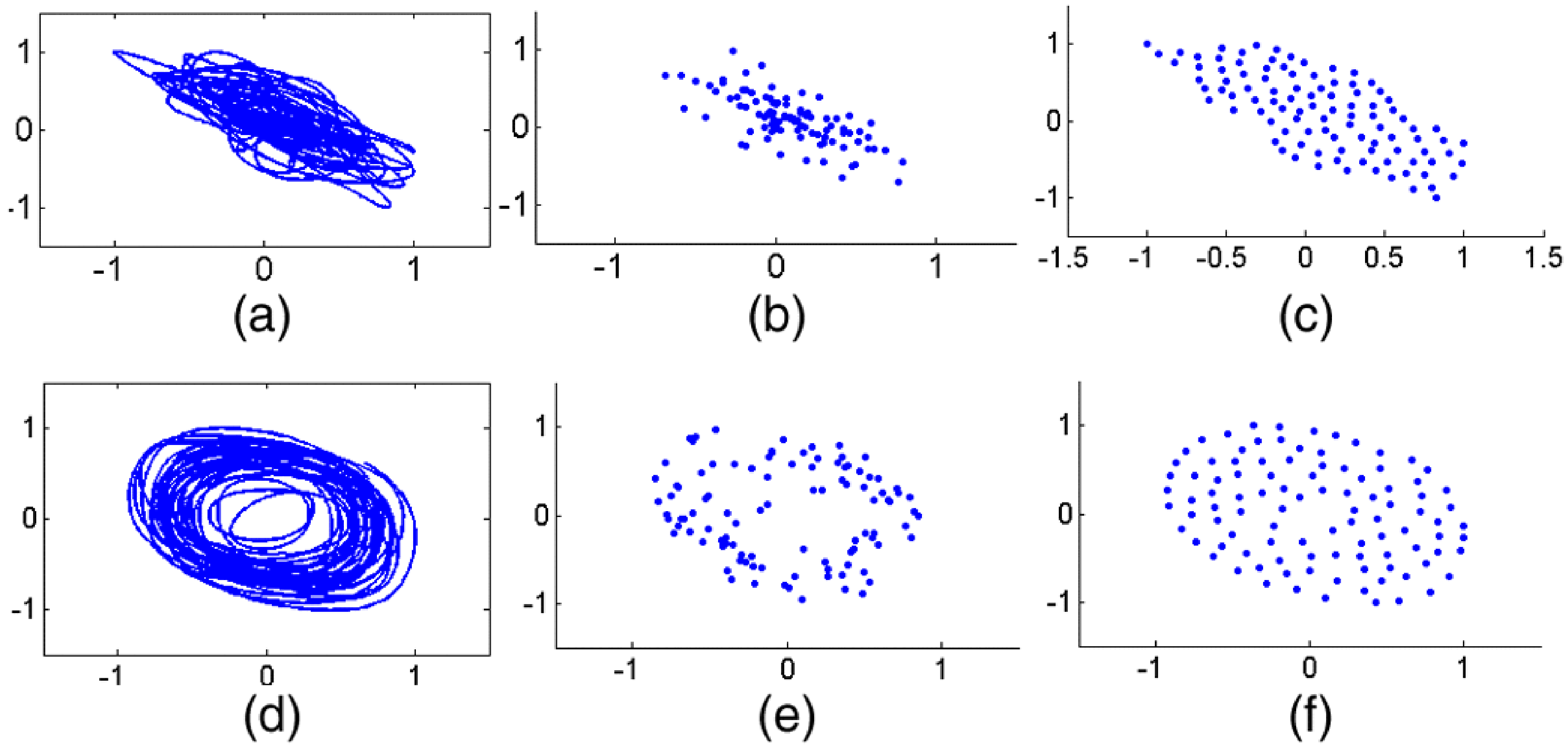


Figure 6.[Emrani Saba, 2014] Point cloud sampling, (a) and (d): the delay embedding of a non-wheeze and a wheeze signal including 4000 points, 100 subsamples slected using random method (b)(e) and maximin method (c)(f).

3.3. Experimental Results

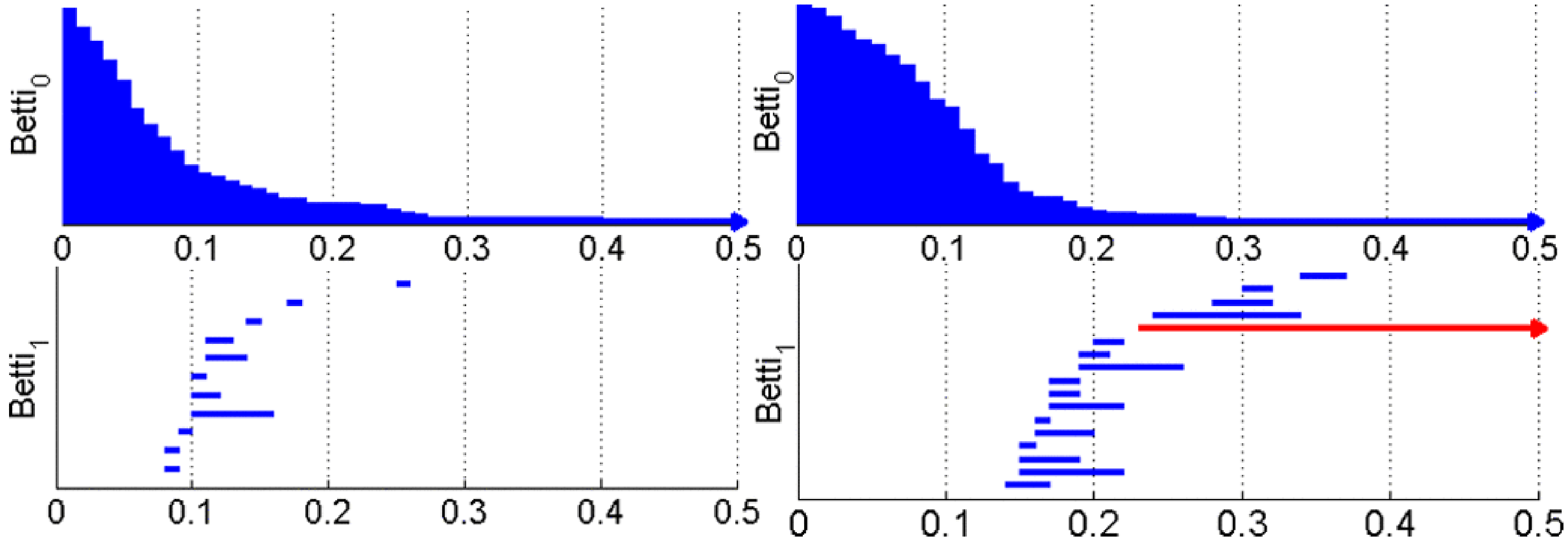


Figure 7.[Emrani Saba, 2014] The barcodes for a non-wheeze breathing sound signal (left) and a wheeze signal (right). The “significant” barcode is highlighted in red and is used to distinguish wheeze signals from non-wheeze signals.

3.3. Experimental Results

- The accuracy of proposed technique is 98.39% while the accuracy of the time-frequency analysis techniques proposed in [3] and [4] without using persistent homology are 86.2% and 95.5%, respectively.

Thanks!