TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

# POWER OPERATIONS OF MORAVA E-THEORY LOCALIZED AT MORAVA K-THEORY

YIFAN WU

Abstract. We calculate $K(n-1)$-localized $E_n$ theory for symmetric groups, deduce the same conclusions as Strickland and find an interpretation of the total power operation $\psi^p_F$ in terms of augmented deformations. Then we specify our calculation to the $n=2$ case. We calculate an explicit formula for $\psi^p_F$ using the formula of $\psi^p_E$, and explain connections between these computations and elliptic curves.

Contents



## 1. $K(n-1)$-localized $E$-theory for symmetric groups

Let $E$ be the Morava E-theory associated to a height $n$ formal group over a field $k$, and $F$ be the $K(n-1)$-localization of $E$. The coefficient ring

$$F^* = W(k)(\!(u_{n-1})\!)^\wedge_p[\![u_1, \ldots, u_{n-2}]\!][u^\pm]$$

is a Noetherian complete local ring with the maximal ideal $(p, u_1, \ldots, u_{n-2})$. It satisfies the conditions in [HKR00, Section 1.3], in particular, $p^{-1}F^* \neq 0$ by direct computation.

In this section, we calculate the ring $F^*B\Sigma_k$ and $F^*B\Sigma_k/I$ following the procedure in [Str98] and give an interpretation of the total power operation $\psi_F$ in terms of subgroups of a certain formal group.

Received by the editors 20th, Dec, 2023.
2010 *Mathematics Subject Classification.* Primary .
*Key words and phrases.* Algebraic Topology, Modular forms.

### 1.1. **Calculations of $F^*B\Sigma_k$ and $F^*B\Sigma_k/I$.**

**Theorem 1.1.** *$F^0B\Sigma_k$ is a Noetherian local ring and a free module over $F^0$ of rank $d(n-1,k)$, which is defined to be the number of isomorphism classes of order $k$ sets with an action of $\mathbb{Z}_p^{n-1}$.*

**Proposition 1.2.** $F^*B\Sigma_k$ is finitely generated over $F^*$.

*Proof.* This is a consequence of [GS99, Corollary 4.4]. We need to verify $F$ is admissible in the sense of [GS99, Definition 2.1]. $E^0$ is Noetherian and both localization and completion preserve Noetherianess. Hence $F^0$ is Noetherian and all other conditions are satisfied automatically. □

**Proposition 1.3.** $F^*B\Sigma_k$ is free over $F^*$, concentrated in even degrees.

*Proof.* From [Str98, Proposition 3.6], we know that $E^*BG$ is concentrated in even degrees. Let $u_{n-1}^{-1}E$ be the homotopy colimit of $\cdots \xrightarrow{u_{n-1}} E \xrightarrow{u_{n-1}} E \to \cdots$, where $u_{n-1}$ is the corresponding element in $E^0$ and let $u_{n-1}^{-1}E/(p, u_1, \ldots, u_{n-2})$ be the cofiber, denoted by $K_{u_{n-1}}$.

We claim that $K^*_{u_{n-1}}B\Sigma_k$ is concentrated in even degrees and free. First $(u_{n-1}^{-1}E)^*B\Sigma_k$ is concentrated in even degrees for the $E^*B\Sigma_k$ being so. Consider the cofibration

$$u_{n-1}^{-1}E \xrightarrow{p} u_{n-1}^{-1}E \to u_{n-1}^{-1}E/(p)$$

which induces a long exact sequence of cohomology groups.

$$0 \longrightarrow (u_{n-1}^{-1}E/p)^{2n-1}B\Sigma_k \longrightarrow (u_{n-1}^{-1}E)^{2n}B\Sigma_k \xrightarrow{p} (u_{n-1}^{-1}E)^{2n}B\Sigma_k \longrightarrow (u_{n-1}^{-1}E/p)^{2n}B\Sigma_k$$

Still from [Str98, Proposition 3.6], the element $p$ acts regularly on $E^{\mathrm{even}}B\Sigma_k$, hence also regular on $(u_{n-1}^{-1}E)^{\mathrm{even}}B\Sigma_k$. Therefore multiplication by $p$ is injective, which implies $(u_{n-1}^{-1}E/p)^*B\Sigma_k$ concentrated in even degrees, then by induction. Since $\pi_*K_{u_{n-1}}$ is a graded field $k(\!(u_{n-1})\!)[u^{\pm}]$, $K^*_{u_{n-1}}B\Sigma_k$ is automatically free.

Now let $F_i = F/(p, u_1, \ldots, u_{i-1})$, and let $F_0 = F$. By construction, we have $F_{n-1} = K_{u_{n-1}}$. We will show that if $F_i^*B\Sigma_k$ is free and concentrated in even degrees, the same is true for $i-1$ as well. Again, there is a long exact sequence of cohomology groups

$$F_{i-1}^*B\Sigma_k \to F_{i-1}^*B\Sigma_k \to F_i^*B\Sigma_k$$

obtained from the cofibration

$$F_{i-1} \xrightarrow{u_i} F_{i-1} \to F_i.$$

Each $F_i^*B\Sigma_k$ is finitely generated by Proposition 1.2. Since $F_i^*B\Sigma_k$ is concentrated in even degrees, multiplying $u_i$ on $F_{i-1}^{\mathrm{odd}}B\Sigma_k$ is a surjective. Hence by Nakayama's lemma, $F_{i-1}^{\mathrm{odd}}B\Sigma_k = 0$. From this, we know the action of $u_i$ on $F_{i-1}^{\mathrm{even}}B\Sigma_k$ is regular, and $F_{i-1}^*B\Sigma_k/u_i = F_i^*B\Sigma_k$ which implies that $F_{i-1}^*B\Sigma_k$ is a free $F^*$ module. □

*Proof of Theorem 1.1.* Applying [HKR00, Theorem C], we have the rank of $p^{-1}F^*B\Sigma_k$ over $p^{-1}F^*$ is just $d(n-1,k)$. By Proposition 1.3, this rank must equal to the rank of $F^*B\Sigma_k$ over $F^*$. □

**Proposition 1.4.** The ring $F^0B\Sigma_k/I = 0$ for $k \neq p^m$ and $R_m := F^0B\Sigma_{p^m}/I$ is a free module over $F^0$ of rank $\overline{d}(n-1,m)$, where $I$ is the transfer ideal and $\overline{d}(n-1,m)$ denotes the number of lattices of index $p^m$ in $\mathbb{Z}_p^{n-1}$.

*Proof.* For the first sentence, there is a standard argument in [Str98, Lemma 8.10]. For the second, using the method in [ST97] we see that $L(DS^0) := \prod L \otimes_{F^0} F^0B\Sigma_k$ is a Hopf ring, which can be identified with the ring of functions $F(\mathbb{B}, L)$, where $L$ is a ring extension of $F^0$ with $p^{-1}$ and all roots of the $p$-series of the formal group law over $F^0$ added and $\mathbb{B}$ is the Burnside semiring.

The $\times-$indecomposables $\mathrm{Ind}L(DS^0) = \prod L \otimes_{F^0} F^0B\Sigma_k/I_k$ is identified with $F(\mathbb{L}, L)$, where $\mathbb{L}$ is the set if all lattices in $\mathbb{Z}_p^{n-1}$ and $I_k$ is the transfer. Hence we have an isomorphism $L \otimes_{F^0} F^0B\Sigma_k/I_k \cong F(\mathbb{L}_k, L)$, with $\mathbb{L}_k$ being the set of such lattices of index $k$. This implies the rank of $R_m$ over $F^0$ is $\overline{d}(n-1,m)$. □

1.2. **Modular interpretation of $\psi_F^p$.** Let $\mathbb{G}_E$ and $\mathbb{G}_F$ be the formal groups over $\mathrm{Spf}(E^0)$ and $\mathrm{Spf}(F^0)$ respectively. In [Str98, Section 9], the scheme $\mathrm{Spf}(E^0B\Sigma_{p^k}/I)$ is identified with the subgroup scheme $\mathrm{Sub}_m(\mathbb{G}_E)$ [Str97, Theorem 10.1] over $\mathrm{Spf}(E^0)$.

The same procedure can be carried through with $E$ replaced by $F$ without harm.

**Proposition 1.5.** There is a canonical isomorphism $\mathrm{Spf}(F^0B\Sigma_{p^m}/I) \to \mathrm{Sub}_m(\mathbb{G}_F)$. That is, the ring $F^0B\Sigma_{p^m}/I$ classifies degree $p^m$ subgroups of $\mathbb{G}_F$.

*Proof.* There is a canonical map from $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G}_F)}$ to $F^0B\Sigma_{p^m}/I$ as constructed in [Str98, Proposition 9.1]. Note that, these two rings has the same rank over $F^0$. So we proceed as [Str98, Theorem 9.2], by showing

$$k(\!(u_{n-1})\!) \otimes_{F^0} \mathcal{O}_{\mathrm{Sub}_m(\mathbb{G}_F)} \to k(\!(u_{n-1})\!) \otimes_{F^0} F^0B\Sigma_{p^m}/I$$

is injective. The key ingredient here is to show $b_m = c_{p^m}^{(p^{n-1}-1)/(p-1)} \neq 0$ in $k(\!(u_{n-1})\!) \otimes_{F^0} F^0B\Sigma_{p^m}$, where $c_{p^m} = e(V_{p^m} - 1)$ is the Euler class of representation $V_{p^m} - 1$ in $F^0B\Sigma_{p^m}$ and $V_{p^m}$ is the standard complex representation of $\Sigma_{p^m}$. To accomplish this, we make a comparison between $E^0B\Sigma_k$ and $F^0B\Sigma_k$.

Let $a_m = c_{p^m}^{(p^n-1)/(p-1)} \in E^0B\Sigma_{p^m}$. It has been shown that $a_m \neq 0 \bmod (p, u_1, \ldots, u_{n-1})$ [Str98, Theorem 3.2]. Consider the diagram

$$\begin{array}{ccc}
E^0B\Sigma_{p^m} & \searrow & \\
\downarrow & & \\
F^0B\Sigma_{p^m} & \longrightarrow & k(\!(u_{n-1})\!) \otimes_{F^0} F^0B\Sigma_{p^m} = K^0_{u_{n-1}}B\Sigma_{p^m}
\end{array}$$

To show $b_m \neq 0$ in the right hand side, it suffices to show the image of $a_m$ in the right corner is not zero. Since $u_{n-1}$ acts regularly on $E^0B\Sigma_{p^m}/(p, u_1, \ldots, u_{n-2})$, we have $a_m \neq 0$ in $u_{n-1}^{-1}E^0B\Sigma_{p^k}$. Otherwise, $u_{n-1}^t a_m = 0$ implies $a_m \in (p, u_1, \ldots, u_{n-2})$.It follows easily that $a_m \neq 0 \bmod (p, u_1, \ldots, u_{n-2})$ in $u_{n-1}^{-1}E^0B\Sigma_{p^m}$. That is

$$a_m \neq 0 \in u_{n-1}^{-1}E^0B\Sigma_{p^k}/(p, u_1, \ldots, u_{n-2}) = K^0_{u_{n-1}}B\Sigma_{p^m}.$$

The rest follows [Str98, Theorem 9.2]. □

*Remark* 1.6. We can not obtain this result directly from [Str97, Theorem 10.1] which asserts that

$$\mathrm{Spf}\, F^0 \times_{\mathrm{Spf}\, E^0} \mathrm{Sub}_m(\mathbb{G}_E) = \mathrm{Sub}_m(\mathrm{Spf}\, F^0 \times_{\mathrm{Spf}\, E^0} \mathbb{G}_E) = \mathrm{Sub}_m(\mathbb{G}_F).$$

The failure of this equation is because the map $E^0 \to F^0$ is not continuous.

In order to figure out how the total power operation

$$\psi_F^p : F^0 \longrightarrow F^0 B\Sigma_p/I$$

interacts with the modular interpretation of $F^0 B\Sigma_p/I$, we shall recall some constructions from [AHS04, Section 3].

Let $Y$ denote the function spectrum $F(\mathbb{C}P^\infty, F)$, we have

$$\pi_0 Y = F^0 \mathbb{C}P^\infty = F^0[\![x]\!]$$

which is a complete local Noetherian ring, with maximal ideal $(p, u_1, \ldots, u_{n-2}, x)$ and the canonical map $\pi_0 F \to \pi_0 Y$ is continuous with respect to their maximal ideal topology.

**Proposition 1.7.** The ring $Y^0 B\Sigma_p/J$ is free over $Y^0$ and equal to $Y^0 \otimes_{F^0} F^0 B\Sigma_p/I$, where $I$ and $J$ are transfer ideals respectively.

*Proof.* For each $k$, we have

$$Y^* B\Sigma_k = [\Sigma_+^\infty B\Sigma_k, F(\mathbb{C}P^\infty, F)] = [\Sigma_+^\infty (B\Sigma_k \wedge \mathbb{C}P^\infty), F] = F^*(B\Sigma_k \wedge \mathbb{C}P^\infty).$$

By the Atiyah Hirzebruch spectral sequence, we have

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, F^q B\Sigma_k) \Rightarrow Y^{p+q} B\Sigma_k$$

Since $F^* B\Sigma_k$ is concentrated in even degrees, we conclude that

$$Y^* B\Sigma_k = Y^* \otimes_{F^*} F^* B\Sigma_k.$$

It follows that $Y^0 \otimes_{F^0} I = J$, and hence

$$Y^0 B\Sigma_p/J = Y^0 \otimes_{F^0} F^0 B\Sigma_p/I.$$

which completes the proof. ∎

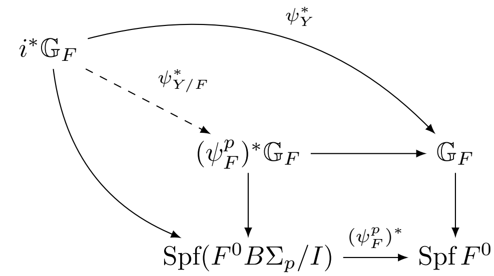
In the language of algebraic geometry, $\operatorname{Spf} Y^0 = \mathbb{G}_F$ and the above proposition can be summarized as the pullback diagram.

$$\begin{array}{ccc} \operatorname{Spf}(Y^0 B\Sigma_p/J) = i^*\mathbb{G}_F & \longrightarrow & \mathbb{G}_F \\ \downarrow & & \downarrow \\ \operatorname{Spf}(F^0 B\Sigma_p/I) & \longrightarrow & \operatorname{Spf} F^0 \end{array}$$

Together with the naturality of the total power operation:

$$\begin{array}{ccc} i^*\mathbb{G}_F & \xrightarrow{\psi_Y^*} & \mathbb{G}_F \\ \downarrow & & \downarrow \\ \operatorname{Spf}(F^0 B\Sigma_p/I) & \xrightarrow{\psi_F^*} & \operatorname{Spf} F^0 \end{array}$$

we obtain a map $\psi_{Y/F}^* : i^*\mathbb{G}_F \to (\psi_F^p)^*\mathbb{G}_F$ over the ring $F^0 B\Sigma_p/I$, as indicated in the diagram.




**Proposition 1.8.** The isogeny $\psi_{Y/F}^* : i^*\mathbb{G}_F \to (\psi_F^p)^*\mathbb{G}_F$ is of degree $p$ over $F^0B\Sigma_p/I$, with kernel the universal degree $p$ subgroup $K$ of $\mathbb{G}_F$ over $F^0B\Sigma_p/I$.

*Proof.* Choosing a coordinate $x$ on $\mathbb{G}_F$, $\psi_Y^*$ sends $x$ to $x^p$ in $Y^0B\Sigma_p/J = \mathcal{O}_{i^*\mathbb{G}_F}$ modulo maximal ideal of $Y^0$. This follows from

$$\pi_0 Y \xrightarrow{D_p} \pi_0 Y^{B\Sigma_p^+} \xrightarrow{S^0 \to B\Sigma_p^+} \pi_0 Y$$

sending $x$ to $x^p$. Since $(\psi_F^p)^*(x) = x$, we conclude that $\psi_{Y/F}^*$ is of degree $p$. Therefore the kernel of $\psi_{Y/F}^*$ is of rank $p$.

To show the kernel is precisely the universal degree $p$ subgroup $K$ of $\mathbb{G}_F$ over $F^0B\Sigma_p/I$, we need to recall the construction of $K$ from [Str98, Proposition 9.1](in which $K$ is denoted by $H_k$). Let $V_p$ be the standard permutation representation of $\Sigma_p$. There is a divisor $\mathbb{D}(V_p)$ of degree $p$ over $F^0B\Sigma_p$, whose base change to $F^0B\Sigma_p/I$ is $K$. Let $A$ be a transitive abelian $p$ subgroup of $\Sigma_p$, we have a composition of maps

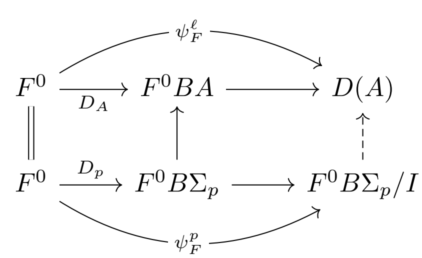
$$\operatorname{Level}(A^*, \mathbb{G}_F) \to \operatorname{Hom}(A^*, \mathbb{G}_F) = \operatorname{Spf} F^0BA \to \operatorname{Spf} F^0B\Sigma_p.$$

The divisor $\mathbb{D}(V_p)$ becomes a subgroup divisor $\Sigma_{a\in A^*}[\ell(a)]$ with $\ell$ the universal level-$A^*$ structure of $\mathbb{G}_F$ on $\operatorname{Level}(A^*, \mathbb{G}_F)$(See [AHS04, Section 3] for definition). It is claimed in [Str98, Proposition 9.1] that the map

$$\operatorname{Level}(A^*, \mathbb{G}_F) \to \operatorname{Spf} F^0B\Sigma_p$$

factors through $\operatorname{Spf} F^0B\Sigma_p/I$ and the union of the images of these maps for all such $A$ is actually $\operatorname{Spf} F^0B\Sigma_p/I$. Hence it is sufficient to show the base change of $\ker \psi_{Y/F}^*$ to $\operatorname{Level}(A^*, \mathbb{G}_F)$ is $\Sigma_{a\in A^*}[\ell(a)]$.

Now Let $D(A) = \mathcal{O}_{\operatorname{Level}(A^*, \mathbb{G}_F)}$, the following diagram




implies the composition of the total power operation $\psi_F^p$ and the dashed arrow is $\psi_F^\ell$ (See [AHS04, Definition 3.9]). Hence after base change to $\operatorname{Level}(A^*, \mathbb{G}_F)$, the map $\psi_{Y/F}^*$ becomes $\psi_\ell^{Y/F}$ [AHS04, diagram 3.14]. According to [AHS04, Proposition 3.21], the kernel of $\psi_\ell^{Y/F}$ is precisely $\ell[A] = \Sigma_{a\in A^*}[\ell(a)]$. □

1.3. **Augmented deformations.** In this section, we combine our analysis about $F^0 B\Sigma_p/I$ and the modular interpretation of $F^0$ in terms of augmented deformations. Recall that there is a formal group $\mathbb{G}_F$ over $F^0$, which is the base change of the universal deformation $\mathbb{G}_E$. Let $\mathbb{G}_F^0$ be the special fiber of $\mathbb{G}_F$, which is the base change of $\mathbb{G}_F$ over the residue field $k(\!(u_{n-1})\!)$ of $F^0$.

The formal group $\mathbb{G}_F^0$ has height $n-1$ over $k(\!(u_{n-1})\!)$. At first glance, one would like to construct the deformation theory of $\mathbb{G}_F^0$ as [LT66] does. However, the problem arises immediately for the field $k(\!(u_{n-1})\!)$ being imperfect. A way to avoid the imperfectness is the treatment stated in [Van21]. We shall recall these constructions.

**Definition 1.9.** An augmented deformation of a formal group $\mathbb{H}$ over $k(\!(u_{n-1})\!)$ consists of a triple $(\mathbb{K}/R, i, \alpha)$ where

- $R$ is a complete local ring and $\mathbb{K}$ is a formal group over $R$,
- A local homomorphism $i : \Lambda \to R$ fits into the commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad i \quad} & R \\ \downarrow & & \downarrow \\ k(\!(u_{n-1})\!) & \xrightarrow{\ \bar{i}\ } & R/\mathfrak{m} \end{array}$$

- and an isomorphism $\alpha : \mathbb{H} \otimes^{\bar{i}}_{k(\!(u_{n-1})\!)} R/\mathfrak{m} \simeq \mathbb{K} \otimes_R R/\mathfrak{m}$,

where $\Lambda = W(k)(\!(u_{n-1})\!)^\wedge_p$ is a Cohen ring with residue field $k(\!(u_{n-1})\!)$.

**Theorem 1.10** ([Van21], Theorem 1.1)**.** *The ring $F^0$ classifies augmented deformations of $\mathbb{G}_F^0$. To be precise, let $\mathrm{Def}^{\mathrm{aug}}_{\mathbb{G}_F^0}(R)$ denote the groupoid of augmented deformations of $\mathbb{G}_F^0$ together with isomorphisms. Then we have*

$$\mathrm{Def}^{\mathrm{aug}}_{\mathbb{G}_F^0}(R) = \mathrm{Maps}_{cts}(F^0, R).$$

*In particular, this implies the moduli problem of classifying augmented deformation is discrete.*

*Proof.* This is simply a consequence of [LT66]. Suppose $\Gamma$ is a height $n$ formal group over a field $k$, with $k$ a residue field of a complete local $A$ algebra. The functor $\mathrm{Def}^A_\Gamma : \mathrm{CLN}_A \to \mathrm{Groupoids}$ from the category of complete local Noetherian $A$ algebras to groupoids which sends $R$ to the groupoid of deformations of $\Gamma$ over $R$ is discrete and corepresented by the ring $A[\![u_1, \ldots, u_{n-1}]\!]$.

Note that for any $R \in \mathrm{CLN}$, with the diagram

$$\begin{array}{ccc} \Lambda & \overset{i}{\dashrightarrow} & R \\ \downarrow & & \downarrow \\ k(\!(u_{n-1})\!) & \xrightarrow{\ \bar{i}\ } & R/\mathfrak{m} \end{array}$$

there is a continuous map from $\Lambda$ to $R$, which lifts $\bar{i}$ [Van21, Corollary 2.9]. Therefore the ring $R$ which carries a deformation of $\mathbb{G}_F^0$ is automatically a $\Lambda$ algebra. Thus we have

$$\mathrm{Def}_{\mathbb{G}_F^0}(R) = \mathrm{Def}^{\Lambda}_{\mathbb{G}_F^0}(R) = \mathrm{Def}^{\mathrm{aug}}_{\mathbb{G}_F^0}(R).$$

Applying the Lubin-Tate's theorem, we find that the functor $\mathrm{Def}^{\Lambda}_{\mathbb{G}_F^0}$ is corepresented by the ring $\Lambda[\![u_1, \ldots, u_{n-2}]\!]$, which is just $F^0$. $\square$

**Theorem 1.11.** *The ring $F^0 B\Sigma_{p^m}/I$ is free over $F^0$ of rank $\overline{d}(m, n-1)$. It classifies augmented deformations of $\mathbb{G}_F^0$ together with a subgroup of degree $p^m$.*

$$\mathrm{Maps}_{cts}(F^0 B\Sigma_{p^m}/I, R) = \{(\mathbb{K}/R, H)\}$$

*To be precise, for any complete local ring $R$, there is a bijection between the set of continuous maps from $F^0 B\Sigma_{p^m}/I$ to $R$ and the set of all pairs $(\mathbb{K}/R, H)$, where $\mathbb{K}$ is an augmented deformations of $\mathbb{G}_F^0$ and $H$ is a degree $p^m$ subgroup of $\mathbb{K}$.*

*Equivalently, $F^0 B\Sigma_{p^m}/I$ classifies augmented deformations of $m'$th Frobenius* [Rez09, Section 11.3]*, with the universal example*

$$\psi^*_{Y/F} : i^*\mathbb{G}_F \to (\psi_F^{p^m})^*\mathbb{G}_F$$

*defined in the Proposition 1.8.*

*Proof.* Combines Proposition 1.7, 1.8 and Theorem 1.10 □

## 2. An explicit Calculation on the $n = 2$ case

Let $E$ be a Morava $E$-theory of height 2 over the field $\overline{\mathbb{F}}_p$, with

$$E^* = W(\overline{\mathbb{F}}_p)[\![u_1]\!][u^{\pm}].$$

Let $F$ be the $K(1)$ localization of $E$, whose coefficients ring is

$$F^* = W(\overline{\mathbb{F}}_p)(\!(u_1)\!)_p^{\wedge}[u^{\pm}].$$

Let $\mathbb{G}_E$ and $\mathbb{G}_F$ be the formal groups over $E^0$ and $F^0$ respectively.

In this section, we give an explicit calculation of the additive total power operation $\psi_F^p$ in terms of the expression of $\psi_E^p$ for the $n = 2$ case.

### 2.1. The formula for $\psi_F^p$.

The naturality of the total power operations gives a diagram:

$$\begin{array}{ccc} E^0 & \xrightarrow{\psi_E^p} & E^0 B\Sigma_p/I \\ \downarrow & & \downarrow t \\ F^0 & \xrightarrow{\psi_F^p} & F^0 B\Sigma_p/J = F^0 \end{array} \tag{2.1}$$

where $I$ and $J$ are the corresponding transfer ideals. The equality on the right corner is because the formal group $\mathbb{G}_F$ is of height 1, hence $F^0 B\Sigma_p/J$ is free of rank $\bar{d}(1,1) = 1$ over $F^0$.

*Remark* 2.1. From now on, we will use $h$ instead of $u_1$ in $E^*$ and $F^*$. This is because when height is 2, the ring $E^0$ can be viewed as the place where the universal deformation of a certain supersingular elliptic curve is defined. The letter $h$ here stands for the Hasse invariant for it being a lift of Hasse invariant.

The map $t$ in the middle is $E^0$ linear. To see this, consider the diagram

$$\begin{array}{ccccc} E^0(\bigvee_{i=1}^{p-1} B\Sigma_i \times B\Sigma_{p-i}) & \xrightarrow{tr_E} & E^0 B\Sigma_p & \longrightarrow & E^0 B\Sigma_p/I \\ \downarrow & & \downarrow & & \downarrow t \\ F^0(\bigvee_{i=1}^{p-1} B\Sigma_i \times B\Sigma_{p-i}) & \xrightarrow{tr_F} & F^0 B\Sigma_p & \longrightarrow & F^0 B\Sigma_p/J \end{array}$$

The maps in the top row are between $E^0$ modules and maps in the bottom can also be viewed as $E^0$ linear maps via $E^0 \to F^0$. Then one can check that the left two vertical maps are $E^0$ linear, which implies $t$ is $E^0$ linear as well.

Now we can deduce the explicit expression of $\psi_F^p$ via the calculation of $\psi_E^p$, which is summarized in the two theorems below.

**Theorem 2.2** ([Zhu19], Theorem A)**.** *After choosing a preferred model for $E$* [Zhu19, Definition 2,23]*, the ring $E^0B\Sigma_p/I$ can be interpreted as*

$$E^0B\Sigma_p/I = W(\overline{\mathbb{F}}_p)[\![h,\alpha]\!]/w(h,\alpha)$$

*with*

$$w(h,\alpha) = (\alpha - p)\,(\alpha + (-1)^p)^p - \left(h - p^2 + (-1)^p\right)\alpha. \tag{2.2}$$

**Theorem 2.3** ([Zhu19], Theorem B)**.** *The image of $h$ under $\psi_E^p$ is*

$$\psi_E^p(h) = \alpha + \sum_{i=0}^{p} \alpha^i \sum_{\tau=1}^{p} w_{\tau+1} d_{i,\tau}, \tag{2.3}$$

*where $w_i$'s are defined to be*

$$w_i = (-1)^{p(p-i+1)} \left[\binom{p}{i-1} + (-1)^{p+1} p \binom{p}{i}\right]$$

*and*

$$d_{i,\tau} = \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\cdots m_{\tau-n}=\tau+i\\ 1\le m_s\le p+1\\ m_{\tau-n}\ge i+1}} w_{m_1}\cdots w_{m_{\tau-n}}.$$

To determine the image of $h \in F^0 = W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$ under $\psi_F^p$, it suffices to determine the image of $\alpha$ in Theorem 2.2 under the map $t$. Then we have

$$\psi_F^p(h) = t \circ \psi_E^p(h)$$

by the diagram 2.1. Since $t$ is an $E^0$ linear map, this requires us to find the solutions of $w(h,\alpha)$ in $F^0$.

**Proposition 2.4.** There is a unique solution $\alpha^*$ of $w(h,\alpha)$ in $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$ with

$$\alpha^* = (-1)^{p+1} p \cdot h^{-1} + \left(1 + (-1)^{p+1}\frac{p(p-1)}{2}\right) p^3 \cdot h^{-3} + \textit{lower terms} \tag{2.4}$$

satisfies

$$w(h,\alpha) = (\alpha - p)(\alpha + (-1)^p)^p - (h - p^2 + (-1)^p)\alpha = 0.$$

Moreover, we have $\alpha^* = 0 \bmod p$.

*Proof.* We write $w(h,\alpha)$ as $w_{p+1}\alpha^{p+1} + w_p\alpha^p + \cdots + w_1\alpha + w_0$, where $w_{p+1} = 1$, $w_1 = -h$, $w_0 = (-1)^{p+1}p$, and

$$w_i = (-1)^{p(p-i+1)} \left[\binom{p}{i-1} + (-1)^{p+1} p \binom{p}{i}\right]$$

for other coefficients.

Since $h$ is invertible in $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$, the equation $w(h,\alpha)=0$ implies

$$\begin{aligned}\alpha &= h^{-1}(\alpha^{p+1}+w_p\alpha^p+\cdots w_2\alpha^2+w_0)\\ &= h^{-1}w_0+\alpha^2(\alpha^{p-1}+w_p\alpha^{p-2}+\cdots+w_2)h^{-1}\\ &= h^{-1}w_0+h^{-3}w_0^2w_2+\mathit{lower\ terms}\end{aligned}$$

Substituting the second equation into itself recursively gives the desired formula for $\alpha^*$ as described in .

This iteration makes sense because the highest term of $\alpha^*$ is $h^{-1}w_0$ and $p|w_0$. Hence each substitution only create a lower terms, which is divided by a higher power of $p$, than current stage. Hence $\alpha^*=\Sigma_k a_k h^{-k}$ and the coefficient $a_k$ satisfies $\lim_{k\to\infty}|a_k|=0$, which implies $\alpha^*$ is indeed an element in $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$.

The uniqueness comes from the following observation. Note that

$$w(h,\alpha)=\alpha(\alpha^p-h) \bmod p.$$

This implies $w(h,\alpha)$ has only one solution 0 in the residue field of $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$. Therefore it also has a unique solution in $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge$, which is $\alpha^*$. □

**Theorem 2.5.** *Let $F$ be a $K(1)$-local Morava $E$-theory at height 2. The total power operation $\psi_F^p$ on $F^0$ is determined by*

$$\psi_F^p(h)=\alpha^*+\sum_{i=0}^{p}(\alpha^*)^i\sum_{\tau=1}^{p}w_{\tau+1}d_{i,\tau}, \tag{2.5}$$

*where*

$$\alpha^*=(-1)^{p+1}p\cdot h^{-1}+\left(1+(-1)^{p+1}\frac{p(p-1)}{2}\right)p^3\cdot h^{-3}+\mathit{lower\ terms}$$

*is the unique solution of*

$$w(h,\alpha)=(\alpha-p)(\alpha+(-1)^p)^p-(h-p^2+(-1)^p)\alpha$$

*in $W(\overline{\mathbb{F}}_p)(\!(h)\!)_p^\wedge\cong F^0$.*

*The other coefficients $w_i$ and $d_{i,\tau}$ are defined in Theorem 2.3.*

*In particular, $\psi_F^p$ satisfies the Frobenius congruence, i.e. $\psi_F^p(h)\equiv h^p \bmod p$.*

*Proof.* The formula 2.5 is obtained by assembling Theorem 2.3 and Proposition 2.4. The last sentence comes from $\psi_F^p\equiv\sum_{\tau=1}^{p}w_{\tau+1}d_{0,\tau} \bmod p$, for $\alpha^*$ being zero after modulo $p$. Also notice that

$$w_i\equiv 0 \bmod p,\ i=0,2,\cdots,p.$$

Therefore

$$\begin{aligned}\psi_F^p(h) &\equiv \sum_{\tau=1}^{p}w_{\tau+1}d_{0,\tau}\equiv d_{0,p}\\ &\equiv \sum_{n=0}^{p-1}(-1)^{p-n}w_0^n\sum_{\substack{m_1+\cdots m_{p-n}=p\\ 1\le m_s\le p+1\\ m_{p-n}\ge 1}}w_{m_1}\cdots w_{m_{p-n}}\\ &\equiv (-1)^p\sum_{\substack{m_1+\cdots m_p=p\\ 1\le m_s\le p+1\\ m_p\ge 1}}w_{m_1}\cdots w_{m_p}.\end{aligned}$$

The only possibility in the last summation is $m_s = 1$, hence

$$\psi_F^p(h) \equiv (-1)^p w_1^p = (-1)^p(-h)^p = h^p \bmod p$$

. □

**Example 2.6.** We calculate these formulas for small $p$.

When $p = 2$, we have

$$\alpha^* = \frac{-2}{h} + \frac{-8}{h^4} + \frac{96}{h^7} + O(h^{-10})$$

and

$$\begin{aligned}\psi_F^2(h) &= h^2 + \alpha^* - h \cdot (\alpha^*)^2 \\ &= h^2 - \frac{6}{h} - \frac{40}{h^4} - \frac{544}{h^7} + O(h^{-10}).\end{aligned}$$

When $p = 3$, we have

$$\alpha^* = \frac{3}{h} + \frac{108}{h^3} - \frac{162}{h^4} + \frac{7857}{h^5} + O(h^{-6})$$

and

$$\psi_F^3(h) = h^3 - 6h^2 - 96h + 594 - \frac{1158}{h} + \frac{14580}{h^2} + \textit{ lower terms.}$$

*Remark* 2.7. In the $p = 3$ case, this power operation formula is different from which in [Zhu14, Section 5.4]. This is because the equation for $\alpha$ in [Zhu14] is not of the form as 2.2, but these two equations are equivalent [Zhu19, Remark 2.25]. In the semi-stable model of Morava $E$-theory [Zhu19, Definition 2.23, Mod.1$^+$], it is required that $\mathrm{Frob}^2 = (-1)^{p-1}[p]$, for instance, [3] in this case. While in [Zhu14], the model used is $\mathrm{Frob}^2 = [-3]$.

*Remark* 2.8. The formula 2.5 relies on the $E_\infty$ structure on $F$. In our analysis, we equipped $F$ with the $E_\infty$ structure induced from $E$ via localization. However, $F$ itself may admit a different $E_\infty$ structure. See [Van21, Section 6].

2.2. **Interaction with elliptic curves.** In this section, we state how these computations interact with elliptic curves and $p$-divisible groups.

Suppose $C$ is a supersingular elliptic curve over a perfect field $k$ with characteristic $p$. The formal group $\widehat{C}$ associated with $C$ is of height 2. Hence we can transport computations in topology to computations on elliptic curves. This is the initial idea of all explicit computations of height 2 Morava $E$ theories. Rezk calculates the $p = 2$ case [Rez08] and Zhu calculates the $p = 3$ case [Zhu14].

To be explicit, let $\mathscr{M}_N$ be the moduli stack of elliptic curves equipped with $\Gamma_1(N)$ structure, i.e. an $N$ torsion point. Over $\mathbb{Z}[1/N]$, the moduli problem of $[\Gamma_1(N)]$ is representable, i.e. $\mathscr{M}_N/\mathbb{Z}[1/n]$ is a scheme. Choose a supersingular locus on $\mathscr{M}_N$, we can produce a height 2 formal group as stated above. Since $C$ is supersinguar, the formal group $\widehat{C}$ equals to the $p$-divisible group $C[p^\infty]$ of $C$. By this, a deformation of $\widehat{C}$ is the same as a deformation of $C[p^\infty]$, which is equivalent to a deformation of $C$ by the Serre-Tate's theorem [Tat67]. Hence we can construct a universal deformation $C_u$ of $C$ defined over $E$, with the formal group $\widehat{C_u}$ being the universal deformation of $\widehat{C}$. Then we construct a corresponding Morava $E$ theory of height 2 associated with $E$, which is also called $E$, via the Landweber exact functor theorem.

To calculate $E^0B\Sigma_p/I$, it suffices to find the place where the universal degree $p$ subgroup $K$ of $C_u$ is defined, for then $K$ is also the universal degree $p$ subgroup of $C_u[p^\infty] = \widehat{C_u}$. This procedure is feasible guaranteed by the moduli problem $\mathscr{M}_p$ is relative representable and hence the simultaneous moduli problem $[\Gamma_1(N)]\times[\Gamma_0(p)]$ is representable by a scheme $\mathscr{M}_{N,p}$ [KM85]. In practice, one usually calculates the coordinates of a point of exact order $p$ to find the explicit expression of $E^0B\Sigma_p/I$ [Rez08, Zhu14], though these calculations are somehow ad hoc for different primes $p$.

*Remark* 2.9. Since in general, an elliptic curve will have $p+1$ subgroups of degree $p$. The moduli scheme $\mathscr{M}_{N,p}$ is of rank $p+1$ over $\mathscr{M}_N$, which is compatible with the rank of $E^0B\Sigma_p/I$ over $E^0$.

*Remark* 2.10. Zhu identifies the parameter $\alpha$ which parametrizes subgroups with a modular form of level $[\Gamma_0(p)]$. He then computes the value of $\alpha$ at cusps of $\mathscr{M}_{N,p}$ and uses this to derived the general formula 2.2 of $E^0B\Sigma_p/I$ for arbitrary primes. [Zhu19]

Recall that the total power operation $\psi^p_E : E^0 \to E^0B\Sigma_p/I$ stands for taking the target of the universal deformation of Frobenius. It can also be viewed as taking the target curve of the universal degree $p$ isogeny as explained above.

Let $\mathscr{C}_N$ be the universal curve of the moduli problem $[\Gamma_1(N)\times[\Gamma_0(N)]$ over $\mathscr{M}_{N,p}$. There is an isogeny $\Psi^p : \mathscr{C}_N \to \mathscr{C}_N/\mathscr{G}^{(p)}_N$, with $\mathscr{G}^{(p)}_N$ the universal degree $p$ subgroup of $\mathscr{C}_N$, $P_0$ the $N$ torsion point:

$$\left(\mathscr{C}_N,\ P_0,\ du,\ \mathscr{G}^{(p)}_N\right) \mapsto \left(\mathscr{C}_N/\mathscr{G}^{(p)}_N,\ \Psi^p(P_0),\ d\tilde{u},\ \mathscr{C}_N[p]/\mathscr{G}^{(p)}_N\right).$$

Hence it induces an exotic endomorphism of $\mathscr{M}_{N,p}$ [KM85, Chapter 11], [Zhu19, Section 2.3], so called *the Atkin Lehner involution*. For a supersingular elliptic curve $S$, this Atkin Lehner involution takes $S$ to itself. Therefore it restricts to an endomorphism of the formal neighborhood around the supersingular locus. The previous argument implies that the total power operation is

$$\psi^p_E : E^0 \hookrightarrow E^0B\Sigma_p/I \xrightarrow{\omega} E^0B\Sigma_p/I,$$

where $\omega$ is the restriction of the Atkin Lehner involution to the formal neighborhood of the given supersingular locus. It is determined by $\psi^p_E(h) = \widetilde{h}$, where $\widetilde{h}$ is the image of $h$ under the Atkin Lehner involution. The calculations along these ideas can be found in [Zhu20, Example 2.14].

Over $F^0$, the $p$-divisible group $\mathbb{G}_E$ becomes an extension

$$0 \to \mathbb{G}_F = \mathbb{G}^0_E \to \mathbb{G}_E \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

where $\mathbb{G}^0_E$ is the connected component of $\mathbb{G}_E$ over $F^0$. Or equivalently

$$0 \to \widehat{C_u} \to C_u[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

over $F^0$. The map $t : E^0B\Sigma_p/I \to F^0$ in 2.1 classifies a degree $p$ cyclic subgroup of $C_u$ over $F^0$. However, in this case, $C_u$ has only one cyclic subgroup of degree $p$, which is compatible with the solution of $w(h, \alpha)$ in $F^0$ being unique, or equivalently, the map $t$ being the unique map from $E^0B\Sigma_p/I$ to $F^0$, as stated in Proposition 2.4. Moreover, this subgroup is also the unique subgroup of degree $p$ of $\widehat{C_u} = \mathbb{G}_F$ over $F^0$.

Therefore, in the interpretation of elliptic curves, we can explain the diagram 2.1 as follow.

$$\begin{array}{ccc} C_u & \xmapsto{\psi_E^p} & C_u/K \\ \downarrow & & \downarrow t \\ C'_u & \xmapsto{\psi_F^p} & C'_u/H \end{array}$$

where $C'_u$ is the base change of $C_u$ over $F^0$, and $H$ is the degree $p$ cyclic subgroup of $C'_u$ explained above. The maps $\psi_E^p$ and $\psi_F^p$ take the target curves of degree $p$ isogenies starting from $C_u$ over $E^0B\Sigma_p/I$ and $F^0$ respectively. And the map $t$ transform $C_u$ to $C'_u$ and $K$ to $H$, hence it takes the curve $C_u/K$ to $C'_u/H$. The element $\psi_F^p(h)$ can be viewed as the Atkin Lehner involution $\widetilde{h}$ restricted over $F^0$.

In the interpretation of formal groups, we have

$$\begin{array}{ccc} \mathbb{G}_E & \xmapsto{\psi_E^p} & (\psi_E^p)^*\mathbb{G}_E = \mathbb{G}_E/K \\ \downarrow & & \downarrow t \\ \mathbb{G}_F & \xmapsto{\psi_F^p} & (\psi_F^p)^*\mathbb{G}_F = \mathbb{G}_F/H \end{array}$$

where $K$ is the universal degree $p$ subgroup of the formal group $\mathbb{G}_E$ and $H$ is the unique degree $p$ subgroup of $\mathbb{G}_F$. The groups $K$ and $H$ are the same thing as which appear in the interpretation of elliptic curves.

*Remark* 2.11. Though the map $t$ takes the universal degree $p$ subgroup $K$ of $\mathbb{G}_E$ to the subgroup $H$ of $\mathbb{G}_F$, we can not conclude this from the Strickland's Theorem [Str97, Theorem 10.1] directly, due to the discontinuity of $t$.

## 3. Connection with Galois Representations

The Cohen ring $\pi_0 L_{K(1)}E_2 = W(k)((u))^\wedge_p$ with residue field $k((u))$ also appears in the $p$-adic galois representation theory over $\mathbb{Z}_p$.

Let $K/\mathbb{Q}_p$ be a finite extension and $K_\infty$ be the maximal cyclotomic extension of $K$.

## References


[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. The sigma orientation is an h-infinity map, 2004.

[GS99] John Greenlees and Neil P. Strickland. Varieties and local cohomology for chromatic group cohomology rings. *Topology*, 38:1093–1139, 1999.

[HKR00] Michael Hopkins, Nicholas Kuhn, and Douglas Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.

[KM85] Nicholas M Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*. Number 108. Princeton University Press, 1985.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal lie groups. *Bulletin de la Société Mathématique de France*, 94:49–59, 1966.

[Rez08] Charles Rezk. Power operations for morava e-theory of height 2 at the prime 2. *arXiv preprint arXiv:0812.1320*, 2008.

[Rez09] Charles Rezk. The congruence criterion for power operations in morava $e$-theory. *Homology, Homotopy and Applications*, 11(2):327–379, 2009.

[ST97] Neil P Strickland and Paul R Turner. Rational morava e-theory and dso. *Topology*, 36(1):137–151, 1997.

[Str97] Neil P Strickland. Finite subgroups of formal groups. *Journal of Pure and Applied Algebra*, 121(2):161–208, 1997.

[Str98] Neil P Strickland. Morava e-theory of symmetric groups. *arXiv preprint math/9801125*, 1998.

[Tat67] John T Tate. p-divisible groups. In *Proceedings of a Conference on Local Fields: NUFFIC Summer School held at Driebergen (The Netherlands) in 1966*, pages 158–183. Springer, 1967.

[Van21] Paul VanKoughnett. Localizations of morava e-theory and deformations of formal groups. *arXiv preprint arXiv:2110.13869*, 2021.

[Zhu14] Yifei Zhu. The power operation structure on morava. *Algebraic & Geometric Topology*, 14(2):953–977, 2014.

[Zhu19] Yifei Zhu. Semistable models for modular curves and power operations for morava e-theories of height 2. *Advances in Mathematics*, 354:106758, 2019.

[Zhu20] Yifei Zhu. The hecke algebra action and the rezk logarithm on morava e-theory of height 2. *Transactions of the American Mathematical Society*, 373(5):3733–3764, 2020.

Southern University of Science & Technology, Shenzhen, China
*Current address*: Department of Mathematics
*Email address*: 12131236@mail.sustech.edu.cn