Pipe Rings and Pipe Formal Groups

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12131236

9th April, 2024

References I

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Contents

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The category **Sch** of schemes does not have all limits and colimits.

A scheme X means a representable functor from **Rings** to **Sets**, that is

$$X = \mathbf{Rings}(A, -) = \mathrm{Spec}(A)$$

for some A. The ring of functions is defined to be

$$\mathcal{O}_X = \mathbb{A}^1(X),$$

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The category of schemes has limits,

$$\lim_i \mathrm{Spec}(A_i) = \mathrm{Spec}(\mathrm{colim}\, A_i)$$

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In general, for two formal schemes $X = \operatorname{colim} X_i$, $Y = \operatorname{colim}_j Y_j$, we define

$$[X, Y] = [\operatorname{colim} X_i, \operatorname{colim} Y_j] = \lim_i [X_i, \operatorname*{colim}_j Y_j] = \lim_i \operatorname*{colim}_j [X_i, Y_j].$$

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There is a kind of special formal schemes, coming from **LRings**.

Suppose R is a linearly topologized ring and S is a ring,

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We denote the full subcategory consisting of solid formal schemes by $\widehat{\mathfrak{X}}_{sol}$.

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Example: $\widehat{\mathbb{A}}^1 = \mathrm{Spf}(\mathbb{Z}[\![t]\!])$, $\widehat{\mathbb{A}}^1(R) = \mathrm{Nil}(R)$.

We say G is a formal group over X if

- $G \cong X \times \widehat{\mathbb{A}}^1$ and
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$f(x, y) = \mu^*(t) \in \mathcal{O}_X[\![x, y]\!]$ is a formal group law.

Suppose $f : G \to H$ is a homomorphism over $X/\mathbb{F}_p$, x, y are coordinates,

$$f^* : \mathcal{O}_X[\![y]\!] \to \mathcal{O}_X[\![x]\!].$$

We have $f^*(y) = g(x^{p^n})$.

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We define $\mathrm{Height}(f)$ to be n in the above equation. $\mathrm{Height}(G)$ is the height of

$$[p] : G \xrightarrow{\Delta} \underbrace{G \times_X \cdots \times_X G}_{p \ \ times} \xrightarrow{\mu} G.$$

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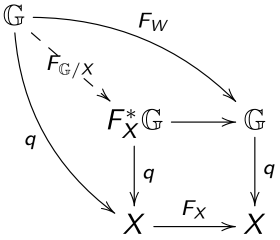
Proposition 1.1

Let $f : \mathbb{G} \to \mathbb{H}$ be a nonzero homomorphism over X with $\mathrm{Height}(\mathbb{G})$ finite. Then $\mathrm{Height}(\mathbb{G}) = \mathrm{Height}(\mathbb{H})$ and $\mathrm{Height}(f)$ is finite.

F_X is the Frobenius.



$\mathbb{G}$: coordinate x, formal group law $g(x, x')$.

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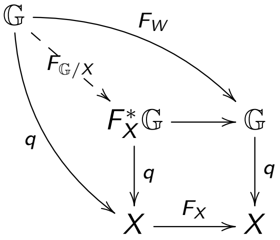

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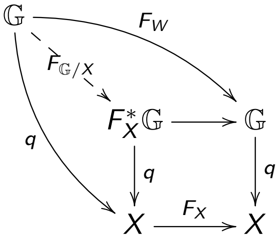

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$$F^*_{\mathbb{G}/X}(y) = x^p$$

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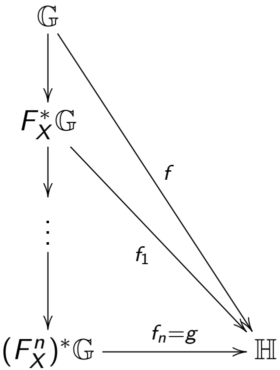
$\mathbb{G}$
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f
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where $I_{h'} = (p, u_1, \cdots, u_{h'-1})$.

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- Inverting topological nilpotent elements destroys the original topology. For instance, inverting x in $k[\![x]\!]$, we have the field $k((x))$.

Goal:

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2. The maps

$$\pi_0 E_h \to \pi_0 L_{K(h')} E_h \to \pi_0 L_{K(h'')} L_{K(h')} E_h \to \cdots$$

belongs to this category.

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$\mathrm{Ind}(C)$ is the category with all filtered colimits added.

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We refer to pipe rings as ring objects in Pipe_∞.

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We define a functor $\mathrm{Pipe}_\infty \to$ **Sets** by

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If R is a pipe ring, then $\underline{R}$ is a ring.

This should be thought as a forgetful functor, which forgets topological structures and continuity of maps.

Every -1-Pipe and 0-Pipe is called fine.

An n-Pipe Y is fine if $Y = \lim_\alpha \mathrm{colim}_\beta (Y_\alpha)_\beta$

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Every -1-Pipe is cofine.
An $n\text{-Pipe}$ X is cofine if $X = \lim_\lambda \operatorname{colim}_\mu (X_\lambda)_\mu$

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Fine and cofine are both preserved by inclusion $\mathrm{Pipe}_{n-1} \to \mathrm{Pipe}_n$.

Proposition 2.1

The realization functor is faithful if the source is cofine and target is fine.

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$$
\begin{aligned}
[X, Y] &= \lim_{\alpha} \operatorname{colim}_{\lambda} \lim_{\nu} \operatorname{colim}_{\beta} [(X_\lambda)_\nu, (Y_\alpha)_\beta] \\
&\subset \lim_{\alpha} \operatorname{colim}_{\lambda} \lim_{\nu} \operatorname{colim}_{\beta} [\underline{(X_\lambda)_\nu}, \underline{(Y_\alpha)_\beta}] \\
&\subset \lim_{\alpha} \operatorname{colim}_{\lambda} \lim_{\nu} [\underline{(X_\lambda)_\nu}, \underline{Y_\alpha}] \\
&= \lim_{\alpha} \operatorname{colim}_{\lambda} [\underline{X_\lambda}, \underline{Y_\alpha}] \\
&\subset \lim_{\alpha} [\underline{X}, \underline{Y_\alpha}] \\
&= [\underline{X}, \underline{Y}].
\end{aligned}
$$

Pipe Dream

For every pipe X, there is an initial cofine pipe X^c over X, such that $X^c \to X$ induces an isomorphism $\underline{X^c} \to \underline{X}$. Dually, for every pipe Y, there is a terminal fine pipe Y^f under Y, which induces $\underline{Y} \to \underline{Y^f}$ an isomorphism. Finally, there is a class of maps W called weak equivalences, such that

$$\mathrm{Pipe}_\infty[W^{-1}](X, Y) = \mathrm{Pipe}_\infty(X^c, Y^f).$$

For $x \in \underline{R}$, we have a map of pipe rings

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$\pi_0 L_{Kh'} E_h$ and its further localizations are bifine.

As in the case of Spec and Spf, for a pipe ring R, we define

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A pipe formal group G over an n pipe R is an $n+1$ pipe, such that $G \cong \widehat{\mathbb{A}}^1_R$, and $\mu : G \times_{\mathrm{Spp}(R)} G \to G$.

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$$\widehat{\mathbb{A}}^2_R = \widehat{\mathbb{A}}^1_R \times_{\mathrm{Spp}(R)} \widehat{\mathbb{A}}^1_R \to \widehat{\mathbb{A}}^1_R$$

By Yoneda lemma, this yields a power series in $\underline{R[\![x_1, x_2]\!]}$.

$$[R[\![x_1, x_2]\!], R[\![x_1, x_2]\!]] \to [R[\![x]\!], R[\![x_1, x_2]\!]]$$
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The pipe formal group over $\pi_0 L_{K(h')} E_h$ has p height h'.
$\pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h$ is bifine.

Staged Lubin-Tate moduli problem: fix $h = h_0 \geq \cdots \geq h_N$

$$R_0 \xrightarrow{i_1} R_1 \to \cdots \xrightarrow{i_N} R_N$$

where R_0 is a complete local ring with residue field k, R_i is an i pipe ring.

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\end{array}$$

F_k is of p height h_k with its associated formal group law pushing forward that of F_{k-1} along $\underline{i_k}$.

Theorem 1

This moduli problem is discrete and corepresented by

$$\pi_0 E_h \to \pi_0 L_{K(h_1)} E_h \to \cdots \to \pi_0 L_{K(h_N)} \cdots L_{K(h_1)} E_h.$$

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Thank You!