§ 0. Double coset formulas.

$\alpha \in GL_2^+(\mathbb{Q})$. Γ_1, Γ_2, congurence of $SL_2(\mathbb{Z})$. Then Γ_1 acts on $\Gamma_1\alpha\Gamma_2$, we have orbit $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2 = \bigcup \Gamma_1\beta_j$. $GL_2^+(\mathbb{Q})$ gurantees that there are finite orbits.

Hence we can define operator $[\Gamma_1\alpha\Gamma_2]_k : M_k(\Gamma_1) \to M_k(\Gamma_2)$ by

$$f[\Gamma_1\alpha\Gamma_2]_k(\tau) = \sum_j f[\beta_j]_k(\tau) = \sum_j \det\beta_j^{k-1} j(\beta_j,\tau)^{-k} f(\beta_j(\tau)).$$

- $f[\Gamma_1\alpha\Gamma_2] \in M_k(\Gamma_2)$: $g \in M_k(\Gamma_2)$ means that $\forall \gamma_2 \in \Gamma_2$,

$$f[\gamma_2]_k(\tau) = j(\gamma_2,\tau)^{-k} f(\gamma_2(\tau)) = f(\tau).$$ Hence

$$(f[\Gamma_1\alpha\Gamma_2]_k)[\gamma_2]_k(\tau) = (\sum_j f[\beta_j]_k)[\gamma_2]_k(\tau)$$

$$= \sum_j j(\gamma_2,\tau)^{-k} f[\beta_j]_k(\gamma_2\tau)$$

$$= \sum_j \det\beta_j^{-k} j(\gamma_2,\tau)^{-k} j(\beta_j,\gamma_2\tau)^{-k} f(\beta_j\gamma_2\tau)$$

$$= \sum_j \det\beta_j\gamma_2^{-k} j(\beta_j\gamma_2,\tau)^{-k} f(\beta_j\gamma_2\tau)$$

$$= \sum_j f[\beta_j\gamma_2]_k(\tau) = \sum_j f[\beta_j]_k(\tau) = f[\Gamma_1\alpha\Gamma_2]_k$$

Since the action of γ_2 on orbit $\{\Gamma_1\beta_j\}$ is just permutation.

Facts : $[\Gamma_1 \alpha \Gamma_2]_k : M_k(\Gamma_1) \to M_k(\Gamma_2)$, sends cusp forms to cusp.

§ 1 Geometric interpretations (in terms of divisors).

- Consider $\alpha = I$. $\Gamma_1 \backslash \Gamma_1\Gamma_2 = \Gamma_1 \cdot \beta_j$, with β_j representative

of $\Gamma_1 \cap \Gamma_2 \backslash \Gamma_2$. [of course $\cup \Gamma_1 \beta_j = \Gamma_1\Gamma_2$. If $\Gamma_1 \cdot \beta_i \cap \Gamma_1 \beta_j \neq \phi$, then

$\gamma_1 \beta_i = \gamma_2 \beta_j \Rightarrow \gamma_2^{-1}\gamma_1 \beta_i = \beta_j \Rightarrow \beta_j \in \Gamma_1 \cap \Gamma_2 \beta_i$. Note that $\gamma_2^{-1}\gamma_1 =$

$\beta_j \beta_i^{-1} \in \Gamma_2$.]

Geometric : For $\Gamma_2 \tau \xmapsto{\text{break}} \{\Gamma_1 \cap \Gamma_2 \beta_j \tau\}_j \xrightarrow{\text{extend}} \{\Gamma_1 \beta_j \tau\}_j$

On modular curves ,

$$\begin{tikzcd}[column sep=small] & \underset{\sum \Gamma_1\cap\Gamma_2 \beta_j \tau}{Y(\Gamma_1 \cap \Gamma_2)} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \underset{\Gamma_2 \tau}{Y(\Gamma_2)} & \text{may ramify} & \underset{\sum \Gamma_1 \beta_j \tau}{Y(\Gamma_1)} \end{tikzcd}$$

$\pi_{1*}\pi_2^* : \mathrm{Div}(X(\Gamma_2)) \to \mathrm{Div}(X(\Gamma_1))$, $\Gamma_2 \tau \longmapsto \sum_j \Gamma_1 \beta_j \tau$

functions : $f \in M_k(\Gamma_1) \xrightarrow{\pi_1^*} \pi_1^* f \in M_k(\Gamma_1 \cap \Gamma_2)$ is natural, consider

as pullback of $\Omega^{\otimes k}$, while on π_2 , one can pushforward a differential

form via tr/Nr (I don't understand)

• Generally, $\alpha \in GL_2^+(\mathbb{Q})$. Consider $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$, two elts are in the same orbit $\Leftrightarrow \gamma(\gamma_1 \alpha \beta_1) = \gamma_2 \alpha \beta_2 \Leftrightarrow \alpha^{-1}\gamma_2^{-1}\gamma\gamma_1 \alpha \beta_1 = \beta_2$.

Hence $\exists\, t \in \alpha^{-1}\Gamma_1 \alpha \cap \Gamma_2$ s.t. $t\beta_1 = \beta_2$. Hence, $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 = \Gamma_1 \alpha \gamma_j$, γ_j's are representatives of $\alpha^{-1}\Gamma_1 \alpha \cap \Gamma_2 \backslash \Gamma_2$. [see DS 5.1.2]

$$\begin{array}{ccccc} \{\Gamma_3 \gamma_j \tau\}_j & \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 & \longrightarrow & \Gamma_1 \cap \alpha\Gamma_2\alpha^{-1} & \{\Gamma_3' \underbrace{\alpha\gamma_j}_{\beta_j} \tau\}_j \\ & \downarrow & & \downarrow & \\ \Gamma_2\tau & \Gamma_2 & & \Gamma_1 & \{\Gamma_1 \beta_j \tau\} \end{array}$$

The horizontal map induces $X(\Gamma_3) \to X(\Gamma_3')$ $\Gamma_3\tau \mapsto \Gamma_3'\alpha(\tau)$

$$\begin{tikzcd} X(\Gamma_3) \arrow[r, "\cdot\alpha"] \arrow[d, "\pi_2"'] & X(\Gamma_3') \arrow[d, "\pi_1"] \\ X(\Gamma_2) & X(\Gamma_1) \end{tikzcd}$$

$\mathrm{Div}(X(\Gamma_2)) \to \mathrm{Div}(X(\Gamma_1))$

$\Gamma_2\tau \longmapsto \sum_j \Gamma_1 \beta_j \tau$

the action on $\Omega^{\otimes k}$, is pullback along $\pi_1 \circ \alpha$ the pushforward along π_2.

§2. Diamond, Atkin-Lehner, Hecke operators

§2.1. Diamond operator $\langle d \rangle = \Gamma_1(N)\alpha\Gamma_1(N)$, $\alpha \in \Gamma_0(N)$ with

$\alpha \equiv \begin{bmatrix} a & b \\ c & \delta \end{bmatrix}$. $\delta \equiv d \bmod N$.

Recall that $\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & b \\ 0 & * \end{bmatrix} \right\}$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$$

$0 \to \Gamma_1(N) \to \Gamma_0(N) \xrightarrow{\pi} (\mathbb{Z}/N\mathbb{Z})^* \to 0$ is exact, $\pi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d \in (\mathbb{Z}/N\mathbb{Z})^*$. Now lets compute β_j's. Consider the diagram

$$\begin{tikzcd} \alpha^{-1}\Gamma_1(N)\alpha \cap \Gamma_1(N) \arrow[r] \arrow[d] & \Gamma_1(N) \cap \alpha\Gamma_1(N)\alpha^{-1} \arrow[d] \\ \Gamma_1(N) & \Gamma_1(N) \end{tikzcd} \qquad \begin{tikzcd} \Gamma_1(N)\tau & \Gamma_1(N)\alpha\tau \\ X_1(N) \arrow[r] \arrow[d] & X_1(N) \arrow[d] \\ X_1(N) & X_1(N) \\ \Gamma_1(N)\tau & \Gamma_1(N)\alpha\tau \end{tikzcd}$$

Hence β_j is just α. Since $\Gamma_1(N)\alpha = \Gamma_1(N)\alpha'$ iff $\pi(\alpha) = \pi(\alpha') = d \in (\mathbb{Z}/N\mathbb{Z})^*$. Hence we define $\langle d \rangle : [\Gamma_1(N)\alpha\Gamma_1(N)]_k$, $\alpha \in \Gamma_0(N)$ with $\alpha = \begin{bmatrix} a & b \\ c & \delta \end{bmatrix}$, $\delta \equiv d \bmod N$.

- $Y_1(N) \xrightarrow{\sim} \{(E, Q) \mid Q \text{ has exact order } n\}/\sim$

$$\begin{tikzcd} Y_1(N) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & \{(E,Q) \mid Q \text{ has exact order } n\}/\sim \arrow[d, leftrightarrow] \\ & \{(E_\tau, \tfrac{1}{N} + \Lambda_\tau)\}/\sim \end{tikzcd}$$

$\alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}$ $\begin{matrix} c \equiv 0 \bmod N \\ \delta \equiv d \end{matrix}$

In this case,

$$\begin{tikzcd} \Gamma_1(N)\tau \arrow[r] \arrow[d] & \Gamma_1(N)\alpha\tau \arrow[d] \\ (E_\tau, \tfrac{1}{N} + \Lambda_\tau) & (E_{\alpha\tau}, \tfrac{1}{N} + \Lambda_{\alpha\tau}) \end{tikzcd}$$

$$\Lambda_{\alpha\tau} = \mathbb{Z}\frac{a\tau+b}{c\tau+\delta} \oplus \mathbb{Z} \cong \Lambda = \mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+\delta) = \Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$$

$$(E_{\alpha\tau}, \tfrac{1}{N} + \Lambda_{\alpha\tau}) \cong \left(E_\Lambda, \frac{c\tau+\delta}{N} + \Lambda\right) = \left(E_\tau, \frac{d}{N} + \Lambda_\tau\right)$$

§ 2.2. Atkin-Lehner. $w_N = [\begin{bmatrix} & -1 \\ N & \end{bmatrix}]_k$. $\Gamma_1(N) \to \Gamma_1(N)$

let $\alpha = \begin{bmatrix} & -1 \\ N & \end{bmatrix}$, then for $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod N$.

$$\alpha^{-1}\beta\alpha = \begin{bmatrix} & \frac{1}{N} \\ -1 & \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} & -1 \\ N & \end{bmatrix} = \begin{bmatrix} d & -\frac{c}{N} \\ -Nb & a \end{bmatrix} \in \Gamma_1(N)$$

Hence $\alpha\Gamma_1(N)\alpha^{-1} \subseteq \Gamma_1(N)$, for the same reason $\alpha^{-1}\Gamma_1(N)\alpha \subseteq \Gamma_1(N)$

$\Rightarrow$ $\alpha^{-1}\Gamma_1(N)\alpha = \Gamma_1(N)$. Diagram

$$\begin{tikzcd}
\Gamma_1(N)\tau & X_1(N) \arrow[r] \arrow[d] & X_1(N) \arrow[d] & \Gamma_1(N)\alpha\tau \\
\Gamma_1(N)\tau & X_1(N) & X_1(N) & \Gamma_1(N)\alpha\tau
\end{tikzcd}$$

Remark: The above calculation also holds for $\Gamma_0(N)$

$$\begin{tikzcd}
\text{Now } \Gamma_1(N)\tau \arrow[r] \arrow[d] & \Gamma_1(N) -\frac{1}{N\tau} \arrow[d] \\
(E_\tau, \frac{1}{N}+\Lambda_\tau) & (E_{-\frac{1}{N\tau}}, \frac{1}{N}+\Lambda_{-\frac{1}{N\tau}}) \arrow[d, phantom, "\cong"] \\
 & (E_{\Lambda'}, \frac{\tau}{N}+\Lambda')
\end{tikzcd}$$

$$\Lambda_{-\frac{1}{N\tau}} = \mathbb{Z}\frac{1}{N\tau} \oplus \mathbb{Z}$$
$$\cong$$
$$\Lambda' = \mathbb{Z}\frac{1}{N} \oplus \mathbb{Z}\tau$$

Note that $\Lambda' \supseteq \Lambda_\tau$ and $\Lambda'/\Lambda_\tau = \langle \frac{1}{N}+\Lambda_\tau \rangle$ is an order n subgroup of E_τ. We can view $E_{\Lambda'} = \mathbb{C}/\Lambda' \cong (\mathbb{C}/\Lambda_\tau)/(\Lambda'/\Lambda_\tau)$ $= E_\tau/\langle \frac{1}{N}+\Lambda_\tau \rangle$. At the meanwhile, $\langle \frac{\tau}{N}+\Lambda' \rangle = E_\tau[N]/\langle \frac{1}{N}+\Lambda_\tau \rangle$

where $E_\tau[N] = \langle \frac{1}{N}+\Lambda_\tau , \frac{\tau}{N}+\Lambda_\tau \rangle$. For $E_\tau[N] = (\mathbb{Z}\frac{\tau}{N} \oplus \mathbb{Z}\frac{1}{N})/\Lambda_\tau$.

$\langle \frac{1}{N}+\Lambda_\tau \rangle = (\mathbb{Z}\tau \oplus \mathbb{Z}\frac{1}{N})/\Lambda_\tau \Rightarrow E_\tau[N]/\langle \frac{1}{N}+\Lambda_\tau \rangle = \langle \frac{\tau}{N}+\Lambda' \rangle$

Remark: The operator w_N is called the Atkin Lehner involution.

For $w_N \circ w_N$: $\Gamma_1(N)\tau \mapsto \Gamma_1(N)\alpha^2\tau = \Gamma_1(N)\begin{bmatrix}-N & 0\\ 0 & -N\end{bmatrix}\tau = \Gamma_1(N)\tau$ is

identity on $X_1(N)$ or $X_0(N)$.

$$\begin{tikzcd}
(E_\tau, \frac{1}{N}+\Lambda_\tau) \arrow[r, mapsto] & (E_{\Lambda'}, \frac{\tau}{N}+\Lambda') \arrow[r, mapsto] & (E_{\Lambda''}, \frac{1}{N^2}+\Lambda'') \arrow[ll, bend left=40] \\
\mathbb{Z}\tau \oplus \mathbb{Z} & \mathbb{Z}\tau \oplus \mathbb{Z}\frac{1}{N} & \mathbb{Z}\frac{\tau}{N} \oplus \mathbb{Z}\frac{1}{N}
\end{tikzcd}$$

§ 2.3 Hecke operators T_p. $\alpha = \begin{bmatrix}1 & \\ & p\end{bmatrix}$. $\Gamma_1(N) \to \Gamma_1(N)$

$\beta_j = \begin{bmatrix}1 & j\\ & p\end{bmatrix}$ $j = 0, 1, \cdots, p-1$ for $p \mid N$.

$\beta_j = \begin{bmatrix}1 & j\\ & p\end{bmatrix}$ $j = 0, 1, \cdots, p-1$. $\beta_\infty = \overset{\in \Gamma_0(N)}{\begin{bmatrix}m & n\\ N & p\end{bmatrix}}\begin{bmatrix}p & \\ & 1\end{bmatrix}$ $p \nmid N$.

$\Gamma_1(N)\tau \xrightarrow{\sim} (E_\tau, \frac{1}{N}+\Lambda_\tau)$, $\Gamma_1(N)\frac{\tau+j}{p}$ corresponds to

$(E_{\frac{\tau+j}{p}}, \frac{1}{N}+\Lambda_{\frac{\tau+j}{p}})$. Now the lattice $\Lambda_{\frac{\tau+j}{p}} = \mathbb{Z}\frac{\tau+j}{p} \oplus \mathbb{Z}$.

need a picture

Note that $\Lambda_\tau \subseteq \Lambda_{\frac{\tau+j}{p}}$ and $\Lambda_{\frac{\tau+j}{p}}/\Lambda_\tau = \langle \frac{\tau+j}{p}+\Lambda_\tau \rangle \subseteq E_\tau$

is a subgroup of order p of E_τ intersect $\langle \frac{1}{N}+\Lambda_\tau \rangle = \{0\}$

So if $p \mid N$, there are exactly p subgroup of order p of E_τ.

they are $\langle \frac{\tau+j}{p} + \Lambda_\tau \rangle$. If $p \nmid N$, there are another one $\langle \frac{1}{p} + \Lambda_\tau \rangle$

In this case, let $\Lambda = \mathbb{Z}\tau \oplus \mathbb{Z}\frac{1}{p}$, $(E_\Lambda, \frac{1}{N} + \Lambda) \cong (E_{p\tau}, \frac{p}{N} + \Lambda_{p\tau})$

In another direction $\Gamma_1(N) \frac{mp\tau+n}{Np\tau+p} \to (E_?, \frac{1}{N} + \Lambda_?) \cong (E_\Lambda, \frac{Np\tau+p}{N} + \Lambda)$

where $\Lambda = \mathbb{Z}(mp\tau+n) \oplus \mathbb{Z}(Np\tau+p) = \Lambda_{p\tau}$ for $\begin{bmatrix} mp\tau+n \\ Np\tau+p \end{bmatrix} = \begin{bmatrix} m & n \\ N & p \end{bmatrix} \begin{bmatrix} p\tau \\ 1 \end{bmatrix}$

and $\begin{bmatrix} m & n \\ N & p \end{bmatrix} \in SL_2(\mathbb{Z})$. Therefore $(E_\Lambda, \frac{Np\tau+p}{N} + \Lambda) = (E_{p\tau}, \frac{p}{N} + \Lambda_{p\tau})$.

To summarize, $T_p : (E, Q) \mapsto \sum_C (E/C, Q+C)$ where Q has exact order N and, C is a order p subgroup of E $\langle Q \rangle \cap C = \{0\}$.

§§ 2.3.1 The intermidieta modular curve. [See DS 1.5.6]

$$\alpha^{-1}\Gamma_1(N)\alpha \cap \Gamma_1(N) = \Gamma_1^0(N,p) = \Gamma_1(N) \cap \Gamma^0(p) \quad \text{, where}$$

$\Gamma^0(p) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \bmod p \in SL_2(\mathbb{Z}) \}$. The corresponding modular curve is $Y_1^0(N,p) \cong \{(E, C, Q) \mid Q \text{ exact order } N,$ C an order p subgroup of E with $\langle Q \rangle \cap C = \{0\}\}$.

On the otherhand, $\Gamma_1(N) \cap \alpha\Gamma_1(N)\alpha^{-1} = \Gamma_{1,0}(N,p) =$

c.f. 7.9.3 D.S.

$\Gamma_1(N) \cap \Gamma_0(Np)$, don't know what it parametrize. $X_{1,0}(N,p)$.

The action of T_p can factor through:

$$
\begin{array}{cccccccc}
(E,Q) & \leftarrow & \sum_C (E,C,Q) & \rightarrow & \boxed{\sum_C (E/C,\ E[p]/C,\ Q)} & \rightarrow & \sum_C (E/C,\ Q) \\
 & & & & \text{my guess} & & \\
X_1(N) & & X_1^0(N,p) & & X_{0,1}(N,p) & & X_1(N) \\
\Gamma_1(N)\tau & & \sum_j \Gamma_1^0(N,p)\,\tau+j & & \sum_j \Gamma_{1,0}(N,p)\,\frac{\tau+j}{p} & & \sum_j \Gamma_1(N)\,\frac{\tau+j}{p}
\end{array}
$$

§ 3. Old forms and New forms.

Why these things matter ?

For $S_k(\Gamma_1(N))$, want to find an orthogonal basis.

Approach : $\{T_n, \langle n\rangle \mid \gcd(n,N)=1\}$ acts on $S_k(\Gamma_1(N))$.

when $\gcd(n,N)=1$, these operators are normal , i.e. $TT^* = T^*T$

Then by standard linear algebra, $S_k(\Gamma_1(N))$ has a basis of simultaneously eigenforms for $\{T_n, \langle n\rangle : (n,N)=1\}$ orthogonal to each other. The limitation comes from $\langle n\rangle, T_n$ are normal when $\gcd(n,N)=1$.

c.f. DS 5.5.2.

Remark : $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha'\Gamma]_k$, $\alpha' = \det(\alpha)\alpha^{-1}$. In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$, then $[\alpha]_k^* = [\alpha']_k \Rightarrow \langle p\rangle^* = \langle p\rangle^{-1}$.

Remark: In any case, $T^* = w_N T w_N^{-1}$, $T = T_n$ or $\langle n\rangle$, without the constrains on $\gcd(n, N)$.

To remove the limitation of $\gcd(n, N) = 1$, it is necessary to introduce newforms. Then $S_k(\Gamma_1(N)) = S_k(\Gamma_1(N))^{new} \oplus S_k(\Gamma_1(N))^{old}$ with eigenforms in new part are eigenforms of all $\{T_n, \langle n\rangle\}$. An old form can be obtained by some new forms in lower level.

§ 3.1 Definitions

If $M | N$, then $\Gamma_1(N) \subseteq \Gamma_1(M)$, hence $S_k(\Gamma_1(M)) \xhookrightarrow{i} S_k(\Gamma_1(N))$

If $d = N/M$, $f \in S_k(\Gamma_1(M))$. Then $f[[\begin{smallmatrix} d & \\ & 1\end{smallmatrix}]]_k(\tau) = d^{k-1} f(d\tau)$ $\in S_k(\Gamma_1(N))$. For $\gamma = [\begin{smallmatrix} a & b \\ x & y\end{smallmatrix}] \equiv [\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}] \bmod N$,

$$d^{k-1} f(d\gamma\tau) = d^{k-1} f\left(\begin{pmatrix} da & db \\ x & y\end{pmatrix}\tau\right) = d^{k-1} f\left(\begin{pmatrix} a & db \\ x/d & y\end{pmatrix} d\tau\right)$$

$$= d^{k-1}(x\tau + y)^{-k} f(d\tau).$$

So we define $i_d : S_k(\Gamma_1(M))^2 \to S_k(\Gamma_1(N))$

$$i_d : (f, g) \mapsto f + g[\alpha_d]_k ,$$

and $S_k(\Gamma_1(N))^{old} := \sum_{d|N} i_d(S_k(\Gamma_1(N/d))^2)$. This definition can be reduced to the equivalent one

$$S_k(\Gamma_1(N))^{old} := \sum_{p|N} i_p(S_k(\Gamma_1(N/p))^2)$$

This is because i_{mn} factors as

$$\begin{array}{ccccc} (S_k(\Gamma_1(N/mn)))^2 & \longrightarrow & (S_k(\Gamma_1(N/m)))^2 & \xrightarrow{i_m} & S_k(\Gamma_1(N)) \\ (f, g) & \longmapsto & (f, g[n]_k) & \longmapsto & f + g[mn]_k \end{array}$$

And we define $S_k(\Gamma_1(N))^{new} = \left(S_k(\Gamma_1(N))^{old}\right)^{\perp}$.

§ 3.2. Properties.

- $S_k(\Gamma_1(N))^{new}$ and $S_k(\Gamma_1(N))^{old}$ are preserved by all T_n, $\langle n \rangle$, $n \in \mathbb{Z}^+$. $\Rightarrow$ orthogonal basis (eigenforms) in each new or old space.

- Criterion ☆ : modify $[\alpha_d]_k : S_k(\Gamma_1(N/d)) \to S_k(\Gamma_1(N))$ by $\iota_d : f(\tau) \mapsto f(d\tau)$. If $f(\tau) = \sum a_n(f) q^n$, then

$l_d f(\tau) = \sum a_n(f) q^{dn}$, $q = e^{2\pi i \tau/n}$. This implies, if $f \in S_k(\Gamma_1(N))$ with $f = \sum_p l_p f_p$ for some $p|N$, $f_p \in S_k(\Gamma_1(N/p))$, then $f(\tau) = \sum a_n(f) q^n$, $a_n(f) = 0$ for all $(n, N) = 1$. The converse is also true: [D.S. 5.7.1][Atkin-Lehner]

If $f \in S_k(\Gamma_1(N))$ with $f(\tau) = \sum a_n(f) q^n$, $a_n(f) = 0$ for all $(n, N) = 1$, then $f = \sum_{p|N} l_p f_p$, $f_p \in S_k(\Gamma_1(N/p))$.

- If f is an eigenform for $\langle T_n, \langle n \rangle, \gcd(n, N) = 1 \rangle$, then $a_n(f) = c_n a_1(f)$. c_n e-value of T_n over f, $\gcd(n, N) = 1$.

So if $a_1(f) = 0 \Rightarrow a_n(f) = 0$ for all $\gcd(n, N) = 1 \Rightarrow f$ is old.

So suppose f is a new eigenform for $(n,N)=1$, then $a_1(f) \neq 0$, assume $a_1(f) = 1$.

Now let $g_m = T_m f - a_m(f) f$ is new. Then $T_n g_m = c_n g_m$ for all $(n, N) = 1$.

But $a_1(g_m) = a_1(T_m f) - a_m f$ = 0 $\Rightarrow g_m$ is old $\Rightarrow g_m = 0$.

" $a_m(f)$ always true

Thus f a new eigenform for $\{T_n, \langle n \rangle; (n, N) = 1\}$ is actually a new eigenform for $\{T_n, \langle n \rangle\}$.

Remark: We also show that if f, g new eigenform, corresponds to the same weight, i.e. $\lambda_f = g_f : T_n \to \mathbb{C}$, then $a_n(f) = c_n a_1(f)$, $a_n(g) = c_n a_1(g) \Rightarrow g = cf$. That is the eigenspace for Hecke alg $\mathbb{T}$ are all 1-dimensional.

Remark: If $f \in S_k(\Gamma_1(N))$ is an eigenform for $\mathbb{T}$, then either f old or new. Clearly if $a_1(f) = 0$, then f is old. If $a_1(f) \neq 0$. $=1$ Then $T_n f = a_n(f) g + a_n(f) h = T_n g + T_n h \Rightarrow T_n g = a_n(f) g$, $T_n h = a_n(f) h$. Do the same with $\langle n \rangle$ shows, g, h are both eigenform for $\mathbb{T}$. If $h \neq 0$, i.e. $a_1(h) \neq 0$, then $T_n h = a_n(h)/a_1(h) \cdot h \Rightarrow$ $a_n(f) = a_n(h)/a_1(h) \Rightarrow f = h/a_1(h)$ is new.