GROUP ACTIONS ON K(1)-LOCALIZED E-THEORY AT HEIGHT 2

YIFAN WU

1. Group actions on height 2 Morava E-theory

Let Γ_2 be the height 2 Honda formal group law over $\mathbb{F}_{p^2}$, which has p series $[p]_\Gamma(x) = x^{p^2}$. Every endomorphism of Γ_2 is defined over $\mathbb{F}_{p^2}$. Let F be the universal deformation of Γ_2 over $W(\mathbb{F}_{p^2})[\![u]\!]$, which has p series

$$[p]_F(x) = px +_F ux^p +_F x^{p^2}.$$

A deformation of Γ_2 consists of a triple $(G/R, i, \alpha)$ with

- G is a formal group law over R,
- $i : \mathbb{F}_{p^2} \to R/m$, an injection into the residue field of R,
- $t : \overline{G} = G \otimes_R R/m \to \Gamma_2 \otimes_{\mathbb{F}_{p^2}} R/m$ an isomorphism.

A $\star$-isomorphism between two such deformations, say (G_1, i, t_1) and (G_2, j, t_2), over R is an isomorphism $f : G_1 \to G_2$ such that the diagram

$$\begin{array}{ccc} \overline{G_1} & \xrightarrow{f} & \overline{G_2} \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} \\ i^*\Gamma_2 & \xrightarrow{j\circ i^{-1}} & j^*\Gamma_2 \end{array}$$

commutes.

Remark 1. *The coefficients of Γ_2 actually lie in $\mathbb{F}_p$, hence the bottom row in previous diagram is in fact the identity map.*

An automorphism α of Γ_2 will induce a permutation on the set of deformations of Γ_2 over R with $\star$-isomorphisms via

$$(G, i, t) \mapsto (G, i, \alpha \circ t),$$

which is represented by a continuous automorphism of $W(\mathbb{F}_{p^2})[\![u]\!]$. Therefore we get an action of the Morava stabilizer group S_2 on $W(\mathbb{F}_{p^2})[\![u]\!]$, with these actions fixing the residue field $\mathbb{F}_{p^2}$.

We can also extend our definitions. Let $\mathbb{G}_2 = \mathrm{Aut}(\mathbb{F}_{p^2}, \Gamma_2)$ be the set of tuples (σ, β) with $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ and $\beta : \sigma^*\Gamma_2 \to \Gamma_2$ being an isomorphism. The group $\mathbb{G}_2$ also acts on the set of deformations, via

$$(G, i, t) \mapsto (G, i \circ \sigma, (i^*\beta)^{-1} \circ t)$$

with the last isomorphism presented by

$$\overline{G} \xrightarrow{t} i^*\Gamma_2 \xleftarrow{i^*\beta} i^*\sigma^*\Gamma_2.$$

These groups fit into a short exact sequence

$$0 \to S_2 = \mathrm{Aut}(\Gamma_2) \to \mathbb{G}_2 \to \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \to 0.$$

2. On K(1)-local height 2 E-theory

Now we turn to the $K(1)$-local case. Let H be the base change of F via the inclusion map $W(\mathbb{F}_{p^2})[\![u]\!] \hookrightarrow W(\mathbb{F}_{p^2})(\!(u)\!)^{\wedge}_p$, which has the p series same as F. Let H_0 be the formal group law by passing to the residue field $\mathbb{F}_{p^2}(\!(u)\!)$. We have

$$[p]_{H_0}(x) = ux^p +_{H_0} x^{p^2}.$$

There are some groups actiong on the ring $\Lambda = W(\mathbb{F}_{p^2})(\!(u)\!)^{\wedge}_p$.

(1) Via the localization map, we have an action of $\mathbb{G}_2$ on Λ. However not all of these actions are continuous. The action of $\mathbb{G}_2$ only ensures that (p, u) is mapped into (p, u) in $W(\mathbb{F}_{p^2})[\![u]\!]$ but it may not take (p) into itself. Hence only a part of $\mathbb{G}_2$ makes sense.

(2) The ring Λ classifies all augmented deformations of H_0 with $\star$-isomorphisms. Recall that an augmented deformation is a triple (G, i, t) with
   - G a formal group law over R,
   - $i : \Lambda \to R$ local homomorphism,
   - $t : \overline{G} \to H_0 \otimes^{\bar{i}}_{\mathbb{F}_{p^2}(\!(u)\!)} R/m$ an isomorphism.

   Following the same procedures, we see that the automorphism group $\mathrm{Aut}(H_0)$ acts continuously on Λ. Since H_0 has height 1, we know that the full automorphism group of H_0 is $S_1 = \mathbb{Z}_p^\times$ over an algebraically closed field.
   Let $\phi(t) = \sum_i b_i t^i$ be an automorphism of H_0, we have

   $$[p]_{H_0}(\phi(x)) = u\phi(x)^p +_{H_0} \phi(x)^{p^2} = \phi(ux^p +_{H_0} x^{p^2}).$$

   Calculating mod x^{p^2}, we see that $u\phi(x)^p = \phi(ux^p)$ mod x^{p^2}. Expanding things out, we have

   $$\sum_{i=1}^{p-1} ub_i^p x^{pi} = \sum_{i=1}^{p-1} b_i u^i x^{pi}.$$

   Comparing coefficients, we find that $b_i^p = b_i u^{i-1}$. In particular $b_1^{p-1} = 1$ means $b_1 \in \mathbb{F}_p$. While if $b_i \neq 0$, we have

   $$b_i^{p-1} = u^{i-1},\ i = 2, 3, \ldots, p-1.$$

   This suggests that we can never expect H_0 to have all automorphisms over any field extension $\mathbb{F}_q(\!(u)\!)$. Therefore there is a subgroup of $\mathbb{Z}_p^\times$ acting on Λ.

(3) From number theory, $p-$adic Galois representations over $\mathbb{Z}_p$, we know there is an action $\mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p^\times$ on Λ, where K is a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^2}$ and $K_\infty = K(\mu_{p^\infty})$ is the maximal cyclotomic extension over K.

So the question here is: How these things fit together?