GOOD GROUPS AND COHOMOLOGY CONCENTRATING IN EVEN DEGREES

YIFAN WU

Abstract. In this note, we review the definition of good groups and the proof of the wreath product lemma in [HKR00]. We will try to extend these ideas from $K(n)$ to an arbitrary field spectrum.

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1. Preliminaries

1.1. **Classifying Spaces.** Let G be a finite group. There is a contractible space EG with free G actions. Denote the quotient space EG/G by BG. We have

$$\pi_n BG = \begin{cases} G, & n = 1 \\ 0, & \text{else} \end{cases}$$

Suppose H is a subgroup of G. Since G acts freely on EG, we have H also acts freely on EG, hence $EG/H = BH$. They fit into a sequence of covering spaces of BG, i.e.

$$EG \longrightarrow EG/H = BH \longrightarrow EG/G = BG.$$

The cover $\pi : BH \to BG$ is $[G : H]$ sheeted with fiber G/H. This map is the B of the inclusion $H \hookrightarrow G$.

On the other hand, we can construct a stable transfer map going on the 'wrong' way. Write $G = \coprod \tau_i H$ and $n = [G : H]$. For each τ_i, left multiplication by g sends it to $\tau_{\sigma(g)(i)} h_{i,g}$. Hence we obtain a permutation representation $G \to \Sigma_n$ and a homomorphism

$$\begin{aligned} G &\to H^n \rtimes \Sigma_n = H \wr \Sigma_n \\ g &\mapsto (h_{1,g}, \dots, h_{n,g}, \sigma(g)) . \end{aligned}$$

We define the *transfer* map to be the composite

$$\begin{tikzcd} BG \arrow[r] \arrow[drrr] & B(H \wr \Sigma_n) \arrow[r, "="] & BH^n \times_{\Sigma_n} E\Sigma_n \arrow[r] & QBH^n \times_{\Sigma_n} E\Sigma_n \arrow[d, "\Theta"] \\ & & & QBH \end{tikzcd}$$

where the map Θ stands for the addition on the infinite loop space QBH. This extends uniquely to a map of spectrum $\mathrm{Tr} : \Sigma_+ BG \to \Sigma_+ BH$.

Remark 1. *In ordinary homology, the transfer map has a neat expression. Suppose Δ is a k-simplex of BG, sufficiently small. Then we have a map $C_*(BG) \to C_*(BH)$*

$$\Delta \mapsto \sum_{\Delta' \in \pi^{-1}\Delta} \Delta'.$$

From this, we see easily that the composite

$$\Sigma_+ BG \xrightarrow{\mathrm{Tr}} \Sigma_+ BH \xrightarrow{\pi} \Sigma_+ BG$$

will induce multiplication by n, the index of the cover, on ordinary homology.

Proposition 1. *Suppose H is a Sylow p-subgroup of G, then the composite*

$$\Sigma_+ BG \xrightarrow{\mathrm{Tr}} \Sigma_+ BH \xrightarrow{\pi} \Sigma_+ BG$$

will induce an equivalence after localizing at (p).

Proof. Because after localization at (p), n is invertible. Hence the composite induces an isomorphism on $\mathbb{Z}_{(p)}$ homology. □

Corollary 1. *In the above situation,* Tr^* *is surjective and* π^* *is injective.*

1.2. **Good Groups.** Let ρ be a complex representation of $H < G$, which is the same as a vector bundle over BH. Let $e(\rho)$ be the corresponding Euler class in $K(n)^*(BH)$.

Definition 1. *We say an element x in $K(n)^*(BG)$ is good, if x is a transferred Euler class of a subrepresentation of G, i.e. $x = \mathrm{Tr}_H^G(e(\rho))$. A group G is good if $K(n)^*(BG)$ is generated by good elements over $K(n)^*$. Of course, $K(n)^*(BG)$ is concentrated in even degrees if G is good.*

Proposition 2. *The following properties for being good hold.*

1. *Every finite abelian group is good.*
2. *G is good if its Sylow p-subgroup is good.*
3. *If x_1 is a good element in $K(n)^*(BG_1)$ and x_2 is good in $K(n)^*(BG_2)$, then so is their product in $K(n)^*(BG_1 \times BG_2)$.*
4. *If $f : K \to G$ is any homomorphism and x is good in $K(n)^*(BG)$, then $f^*(x)$ is a linear combination of good elements in $K(n)^*(BH)$.*
5. *If x and y are both good, then their cup product xy is a sum of good elements.*

Proof.

1. We only need to consider p-components of G, then reduce the case to $G = \mathbb{Z}/p$. While $K(n)^*(B\mathbb{Z}/p) = K(n)^*[x]/[p]_F(x)$ and x is the Euler class of any line bundle corresponding to a generator of the character group $\mathbb{Z}/p^*$.($\alpha : \mathbb{Z}/p \to S^1$ will induce a map $B\mathbb{Z}/p \to BS^1 = \mathbb{C}P^\infty$, and x is the Euler class of the corresponding line bundle.)
2. The map $\mathrm{Tr}^* : K(n)^*(BG_p) \to K(n)^*(BG)$ is surjective.
3. Suppose $x_1 = \mathrm{Tr}_{H_1}^{G_1}(e(\rho_1))$ and $x_2 = \mathrm{Tr}_{H_2}^{G_2}(e(\rho_2))$. We have

$$x_1 \times x_2 = \mathrm{Tr}_{H_1 \times H_2}^{G_1 \times G_2}(e(\rho_1 \oplus \rho_2))$$

.

(4) Suppose $x = \mathrm{Tr}_H^G(e(\rho))$. We have

$$\begin{tikzcd} \prod BK_\alpha \arrow[r, "\prod f_\alpha"] \arrow[d] & BH \arrow[d] \\ BK \arrow[r, "Bf"] & BG \end{tikzcd}$$

The naturality of transfer maps yield

$$f^*(x) = \sum \mathrm{Tr}(e(f_\alpha^*(\rho))).$$

(5) If x and y are good, then $x \times y$ is good in $K(n)^*(B(G \times G))$. Composing with the diagonal $\Delta : G \to G \times G$ gives the cup product xy.

□

Remark 2. *Not all groups are good. In fact, for each p, there are examples of groups which are not good. For $p > 2$, $n \geq 2$, the Sylow p-subgroup $P = (\mathbb{Z}/p)^4 \rtimes (\mathbb{Z}/p)^2$ of $GL_4(\mathbb{F}_p)$ works. See* [KL00] *for detail calculations.*

2. The Wreath Product Lemma

To show a group G is good, we may consider its Sylow p-subgroups. In practice, a lot of such groups have wreath product expressions. For example, Sylow p-subgroups of $B\Sigma_k$ is a (product) of iterated wreath product of $\mathbb{Z}/p$ with itself. Thus it is wonderful if the following is true.

Theorem 1 (The Wreath Product Lemma)**.** *If G is good, then so does the wreath product $G \wr \mathbb{Z}/p$.*

Let W denote $G \wr \mathbb{Z}/p$. Consider the sequence

$$1 \to G^p \to W \to \mathbb{Z}/p \to 1$$

which induces a fiber sequence

$$BG^p \to BW \to B\mathbb{Z}/p.$$

We have the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}(BW) = H^*(B\mathbb{Z}/p, K(n)^*(BG^p)) \Rightarrow K(n)^*(BW)$$

The action of $\mathbb{Z}/p$ over G^p is a cyclic permutation, hence $\mathbb{Z}/p$ acts on $K(n)^*(BG^p) = \otimes K(n)^*(BG)$ by permutation too. Since $K(n)^*(BG)$ is finitely generated, we can choose a basis $\{x_i\}$ of $K(n)^*(BG)$, then

$$K(n)^*(BG^p) = F \oplus T$$

The module F is a free $\mathbb{Z}/p$ module, with basis $\{x_{i_1} \otimes \cdots \otimes x_{i_p}\}$ such that not all i_j are same. The module T has trivial $\mathbb{Z}/p$ action. Therefore, the E_2 page can be identified with

$$E_2 = H^*(B\mathbb{Z}/p, F \oplus T) = H^*(\mathbb{Z}/p, F) \oplus H^*(\mathbb{Z}/p, T).$$

A simple calculation implies

$$H^*(\mathbb{Z}/p, F) = \begin{cases} F^{\mathbb{Z}/p}, & * = 0 \\ 0 \quad , & \text{else} \end{cases}$$

and

$$H^*(\mathbb{Z}/p, T) = H^*(B\mathbb{Z}/p) \otimes T$$

with

$$H^*(B\mathbb{Z}/p) = E(u) \otimes P(x)$$

where $|u| = 1$ and $|x| = 2$.

From above analysis, we find that the part $E_2^{\geq 1,*}$ of the E_2 page is $H^{\geq 1}(B\mathbb{Z}/p) \otimes T$. Since we alredy know that the spectral sequence

$$E_2^{*,*}(B\mathbb{Z}/p) = H^*(B\mathbb{Z}/p, K(n)^*) \Rightarrow K(n)^*(B\mathbb{Z}/p) = K(n)^*[x]/v_n x^{p^n},$$

the only nonzero differential is

$$d_{2p^n-1}(u) = v_n x^{p^n}.$$

Hence we conclude that for $r \geq 2$, the $E_r^{\geq 1,*}(BW)$ page is isomorphic to $E_2^{\geq 1,*}(B\mathbb{Z}/p) \otimes T$. In particular, when $r \geq 2p^n$, there are no differentials in this area.

Lemma 1. *The elements in $E_2^{0,*}(BW)$ are all permanent cycles, which are linear combinations of good elements.*

If this lemma is correct, which means there are also no differentials starting from the 0th column, and then we have

$$E_r^{*,*}(BW) = H^0(\mathbb{Z}/p, F) \oplus (E_r^{*,*}(B\mathbb{Z}/p) \otimes T)\,,$$

and

$$K(n)^*(BW) = F^{\mathbb{Z}/p} \oplus (K(n)^*(B\mathbb{Z}/p) \otimes T)\,.$$

The last indentity implies W is a good group directly.

Proof of Lemma 1. The proof falls into two parts.

An element in $F^{\mathbb{Z}/p} \subset K(n)^*(BG^p)$ is a permanent cycles if and only if it is an image of the restriction map $K(n)^*(BW) \to K(n)^*(BG^p)$. Note that $F^{\mathbb{Z}/p}$ is generated by $\sigma(x) = \sum_{\sigma_i \in \mathbb{Z}/p} \sigma_i(x)$, $x \in K(n)^*(BG^p)$, i.e. the sum of orbits of x. The composite

$$K(n)^*(BG^p) \xrightarrow{\mathrm{Tr}} K(n)^*(BW) \xrightarrow{\mathrm{Res}} K(n)^*(BG^p)$$

will send x to $\sigma(x)$.

An element in T is of the form $x \otimes \cdots \otimes x$ for $x \in K(n)^*(BG)$. We can assume $x = \mathrm{Tr}_H^G(e(\rho))$ is a transferred Euler class. The representation $\rho \oplus \cdots \oplus \rho$ is a representation of H^p, which extends to a representation $\widehat{\rho}$ of $H \wr \mathbb{Z}/p$. The result follows from the diagram.

$$\begin{tikzcd}
K(n)^*(BH^p) \arrow[d, "\mathrm{Tr}"] & K(n)^*(B(H \wr \mathbb{Z}/p)) \arrow[l, "\mathrm{Res}"'] \arrow[d, "\mathrm{Tr}"] \\
K(n)^*(BG^p) & K(n)^*(BW) \arrow[l, "\mathrm{Res}"']
\end{tikzcd}$$

$\square$

3. Generalizing to Arbitrary Field Spectra

In the above discussions, we realize that a lot of properties does not rely on the cohomology theory $K(n)$. It is natural to ask if we can extends our definition of good groups to a general setting.

Let F be an even periodic field spectrum, i.e. a ring spectrum with

$$\pi_* F = k[\beta^{\pm}]$$

for some β in degree 2 and $\operatorname{char} k = p$. F is automatically complex oriented. We denote the formal group law over F still by F, and its height by n. Under these settings, the spectrum F enjoys some significant properties.

Proposition 3. *There is a linearly duality between F-homology and cohomology, i.e.*

$$F^*(X) = \mathrm{Hom}_{F_*}(F_*(X), F^*).$$

For spectra X and Y, we have a Künneth homeomorphism:

$$F^*(X)\widehat{\otimes}F^*(Y) \xrightarrow{\sim} F^*(X \wedge Y)$$

Proof. There are two ways to see the first statement. One way is applying the universal coefficient spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{F_*}^{*,*}(F_*(X), F^*) \Rightarrow F^*(X).$$

Since all things are free F_* modules. The E_2 page collapses and we only has the 0th column, i.e. the Hom part.

The second way is to look at the Serre spectral sequences. The homological and cohomological spectral sequences are dual to each other (both terms and differentials), which yields the conclusion.

The second statement is [Boa95, Theorem 4.19]. □

Proposition 4. *For each finite group G, $F^*(BG)$ is finite as F^* modules.*

Proof. This is proved for $F = K(n)$ in [Rav82] of which I haven't found the citation link. We will recall his proof in our settings.

First, we may assume G is a p-group, for the surjectivity of transfer maps. We can find a normal subgroup H of G with index p, and a group $\widehat{G}$ with $\widehat{G}/H \cong \mathbb{Z}$.

Assume $F^*(BH)$ is finite. The fiber sequence

$$BH \to B\widehat{G} \to S^1$$

implies $F^*(B\widehat{G})$ is finite.

Consider the map between fiber sequences

$$\begin{tikzcd} B\widehat{G} \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d, "\mathrm{Id}"] \\ S^1 \arrow[r] & B\mathbb{Z}/p \arrow[r] & \mathbb{C}P^\infty \end{tikzcd}$$

The Atiyah Hirzebruch spectral sequence for the bottom row implies there is a differential killing x^{p^n} for $F^*(B\mathbb{Z}/p) = F^*[x]/g(x)$, where $g(x)$ is the degree p^n Weierstrass polynomial associated to $[p]_F(x)$.

Finally, we see that $E_r^{*,*}(BG)$ is a module over $E_r^{*,*}(B\mathbb{Z}/p)$. Hence x^{p^n} is killed in $E_r^{*,*}(BG)$. The finiteness of $F^*(BG)$ follows from it of $F^*(B\widehat{G})$. □

Now we can try to generalize the theory of good groups to our settings.

Definition 2. *Let G be a finite group. We say an element $x \in F^*(BG)$ is F-good and G is F-good if they satisfy the conditions in Definition 1 with $K(n)$ replaced by F.*

Proposition 5. *Proposition 2 still holds for F-goodness.*

Proposition 6. *The wreath product lemma holds for F-goodness.*

Proof. All things follow from the proof of Theorem 1. □

Remark 3. *It says in* [Lur10, Lecture 24, Proposition 9,10] *that F is a $K(n)$-module. Hence F is equivalent to $\bigvee_{\alpha_k} \Sigma^{\alpha_k} K(n)$ with all α_k even. There are two questions.*

(1) *How to express for example $K_{u_{n-1}} := L_{K(n-1}E_n/(p, u_1, \dots, u_{n-2})$ as a wedge sum of $K(n-1)$, where $K^*_{u_{n-1}} = \mathbb{F}_p(\!(u_{n-1})\!)[\beta^{\pm}]$ with $|\beta| = 2$.*
(2) *What's the relation between $K(n)^*(X)$ and $M^*(X)$ where $M = \bigvee_{\alpha_k} \Sigma^{\alpha_k} K(n)$.*

If $M^*(X)$ behaves as what one hopes, then we can directly say that $F^*(BG)$ is concentrated in even degrees.

Corollary 2. *$F^* B\Sigma_k$ is concentrated in even degrees.*

References

[Boa95] J Michael Boardman. Stable operations in generalized cohomology. *Handbook of algebraic topology*, pages 585–686, 1995.

[HKR00] Michael Hopkins, Nicholas Kuhn, and Douglas Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.

[KL00] Igor Kriz and Kevin P Lee. Odd-degree elements in the morava k (n) cohomology of finite groups. *Topology and its Applications*, 103(3):229–241, 2000.

[Lur10] Jacob Lurie. Chromatic homotopy theory. *Lecture series*, 2010.