FORMAL SCHEMES AND FORMAL GROUPS

YIFAN WU

Abstract. This note is unfinished. It consists of the basic notions and ideas of formal schemes and formal groups. The theory of Level structures is only mentioned, but not finished.

1. Formal schemes

In algebraic geometry, a formal scheme is used to detect the local behavior around a closed point. For example, Let $R = k[x]$ for some field k. The maximal ideal (x) corresponds to the closed point $[0]$. To study the local behavior around this closed point, one has a sequence

$$\mathrm{Spec}(k) \cong \mathrm{Spec}(k)[x]/x \to \mathrm{Spec}(k)[x]/x^2 \to \cdots \to \mathrm{Spec}(k)[x]/x^n \to \cdots .$$

In each stage, Spec $k[x]/x^n$ has the underlying space $[0]$, but the functions are more. This indicates us to take the colimit of this sequence. Unfortunately, the category **Sch** of schemes does not have all limits and colimits.

Remark 1. *The category of locally ringed spaces has all limits and colimits. The category* **Aff** *of affine schemes is the opposite category of* **Ring**. *We have the adjunction*

$$\Gamma : \mathbf{Sch} \leftrightarrows \mathbf{Ring}^{op} : \mathrm{Spec}$$

with Spec *being a right adjoint. Hence it preserves the limits in* $\mathbf{Ring}^{op}$, *or equivalently, colimits in* **Ring**.

No matter in what cases, there is no evidence that the colimit should exist. Hence we have the following definition. Now we say a scheme means an affine scheme in all of the notes, and denote the category of affine scheme by $\mathfrak{X}$, the full subcategory of $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$ consisting of representable functors.

Definition 1. *A formal scheme X is a small filtered colimit of scheme X_i. As we already explained, this colimit may not exist in $\mathfrak{X}$. We can embed $\mathfrak{X}$ into $\mathbf{Fun}(\mathbf{Ring}, \mathbf{Set})$, where the later always has colimits, pointwisely.*

To be more concrete, for each ring R, we have

$$X(R) = \mathrm{colim} X_i(R).$$

Definition 2. *Let $X = \mathrm{colim} X_i$ and $Y = \mathrm{Y_j}$ be formal schemes. Define*

$$\widehat{\mathfrak{X}}(X, Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i, Y_j).$$

We denote $\widehat{\mathfrak{X}}(X, \mathbb{A}^1)$ by $\mathcal{O}_X$, where $\mathbb{A}^1 = \mathbf{Ring}(\mathbb{Z}[t], -)$. To be precise,

$$\mathcal{O}_X = \lim \mathcal{O}_{X_i}.$$

Remark 2. *From the definition, $\widehat{\mathfrak{X}}$ is actually the same as $\mathcal{I}nd(\mathfrak{X})$. Note that in general, one has*

$$[\operatorname{colim} X_i, Y] = \lim[X_i, Y].$$

By the definition of $\operatorname{colim} Y_j$*, we have*

$$\widehat{\mathfrak{X}}(X,Y) = \lim_i \widehat{\mathfrak{X}}(X_i,Y) = \lim_i \operatorname*{colim}_j \mathfrak{X}(X_i,Y_j).$$

This is how we define morphisms in $\widehat{\mathfrak{X}}$.

Example 1. *Let $N_i = \operatorname{Spec} \mathbb{Z}[x]/x^n$. The resulting formal scheme is denoted by $\hat{\mathbb{A}}^1$. Note that $\hat{\mathbb{A}}^1(R) = \operatorname{colim}\mathbf{Ring}(\mathbb{Z}[x]/x^n, R)) = \operatorname{Nil}(R)$. And $\mathcal{O}_{\hat{\mathbb{A}}^1} = \mathbb{Z}[\![x]\!]$.*

The category $\widehat{\mathfrak{X}}$ has better categorical properties than $\mathfrak{X}$.

(1) $\widehat{\mathfrak{X}}$ has all small colimits and finite limits.
(2) finite limits commute with small colimits in $\widehat{\mathfrak{X}}$.

There are a special kind of formal schemes, called *solid* formal scheme, $\operatorname{Spf}(R)$, which we will define right now.

Definition 3. *A linear topologized ring is a ring R equipped with a neighborhood system around 0 consisting of open ideals, which forms a topological basis under translation. The category of such rings and continuous maps is denoted by* **LRing***.*

We can equip any ring S with discrete topology, which yields a fully faithful embedding **Ring** $\to$ **LRing**. Suppose $R \in$ **LRing**, $S \in$ **Ring**, f is a continuous map from R to S. We must have $f^{-1}(0) = J$ an open ideal in R. Hence f is equivalent to a map $R/J \to S$ between rings. All open ideals in R form a cofiltered system under inclusion maps. Hence we have

$$\mathbf{LRing}(R,S) = \operatorname*{colim}_J \mathbf{Ring}(R/J,S).$$

Therefore we define $\operatorname{Spf}(()R) \in \mathbf{Fun}(\mathbf{Ring},\mathbf{Set})$ by

$$\operatorname{Spf}(R)(S) = \operatorname*{colim}_J \mathbf{Ring}(R/J,S).$$

Definition 4. *A solid formal scheme is a formal scheme which is isomorphic to* $\operatorname{Spf}(R)$ *for some linearly topologized ring R. The solid formal schemes form a full subcategory $\widehat{\mathfrak{X}}_{sol}$ of $\widehat{\mathfrak{X}}$.*

Given a linearly topologized ring R, we have the related cofiltered system $\{R/J\}$, where J runs through all open ideals. The limit of this system is denoted by $\widehat{R}$, called the completion of R. The ring $\widehat{R}$ automatically inherits a topological structure from R. The preimage $\overline{J}$ of J under the natural map $\widehat{R} \to R/J$ forms a neighborhood system around 0 in $\widehat{R}$. It is easy to check $\operatorname{Spf}(\widehat{R}) = \operatorname{Spf}(R)$. A ring R is complete or a formal ring if $R = \widehat{R}$. The category of formal rings is denoted by **FRing**, which is a full subcategory of **LRing**.

Note that

$$\widehat{\mathfrak{X}}(X,\operatorname{Spf}(R)) = \lim_i \widehat{\mathfrak{X}}(X_i,\operatorname{Spf}(R)) = \lim_i \mathbf{LRing}(R,\mathcal{O}_{X_i}) = \mathbf{LRing}(R,\mathcal{O}_X).$$

Hence we have the adjoint pairs:

$$\mathcal{O} : \widehat{\mathfrak{X}} \leftrightarrows \mathbf{LRing}^{op} : \operatorname{Spf}.$$

We have the unit map $X \to \operatorname{Spf}(\mathcal{O}_X)$, and the counit $R \to \widehat{R}$, which is just the completion.

Proposition 1. *We have the following propositions.*

1. *X is a solid fomal scheme then $\mathcal{O}_X$ is a formal ring.*
2. *X is solid iff $X \to \mathrm{Spf}(\mathcal{O}_X)$ is an isomorphism.*
3. *The inclusion functor $\widehat{\mathfrak{X}}_{sol} \to \widehat{\mathfrak{X}}$ is right adjoint to $X \to \mathrm{Spf}(\mathcal{O}_X)$.*
4. *The inclusion functor $\mathbf{FRing} \to \mathbf{LRing}$ is right adjoint to taking completion.*

Proof. (1) Obvious.
(2) X is solid, then X is isomorphic to $\mathrm{Spf}(R)$ for some R. Therefore $\mathcal{O}_X$ is isomorphic to $\widehat{R}$, which yields the conclusion. The converse is obvious.
(3) The functor $\mathbf{FRing}^{op} \to \widehat{\mathfrak{X}}_{sol}$ sending R to $\mathrm{Spf}(R)$ is fully faithful. Suppose R, S are two formal rings, then

$$\widehat{\mathfrak{X}}_{sol}(\mathrm{Spf}(R), \mathrm{Spf}(S)) = \lim_J \widehat{\mathfrak{X}}_{sol}(\mathrm{Spec}(R/J), \mathrm{Spf}(S)) = \lim \mathbf{LRing}(S, R/J) = \mathbf{FRing}(S, R).$$

Therefore by (2), this functor defines an equivalence. The equation

$$\widehat{\mathfrak{X}}(X, \mathrm{Spf}(R)) = \mathbf{LRing}(R, \mathcal{O}_X) = \widehat{\mathfrak{X}}_{sol}(\mathrm{Spf}(\mathcal{O}_X), \mathrm{Spf}(R))$$

implies the inclusion functor being right adjoint to $X \to \mathrm{Spf}(\mathcal{O}_X)$.
(4) The same argument holds.

$$\mathbf{LRing}(R, \widehat{S}) = \mathrm{Spf}(R)(\widehat{S}) = \mathrm{Spf}(\widehat{R})(\widehat{S}) = \mathbf{FRing}(\widehat{R}, \widehat{S}).$$

□

Definition 5. *Suppose $R \to S$ and $R \to T$ are continuous maps in $\mathbf{LRing}$. Define their tensor products $S \otimes_R T$ to be the usual tensor product equipped with linear topology spanned by $S \otimes_R I + J \otimes_R T$, for open ideals $I \subset T$ and $J \subset S$. This actually the pushout in $\mathbf{LRing}$. If both of them are formal rings, then we define $S \widehat{\otimes}_R T$ to be the completion of $S \otimes_R T$, which corresponds to the pushout in $\mathbf{FRing}$ for the completion being a left adjoint.*

2. Formal Groups

Definition 6. *A formal group G over a formal scheme X is a group object in $\widehat{\mathfrak{X}}_X$. We also require that G is isomorphic to $X \times \widehat{\mathbb{A}}^1$ in $\widehat{\mathfrak{X}}_X$. A map $u : G \to \widehat{\mathbb{A}}^1$ makes G isomorphic to $X \times \widehat{\mathbb{A}}^1$ is called a coordinate on G.*

Suppose X is solid. Then $X \times \widehat{\mathbb{A}}^1$ is again solid. From the equivalence of categories, we have $X \times \widehat{\mathbb{A}}^1$ is isomorphic to the Spf of coproduct of $\mathcal{O}_X$ and $\mathbb{Z}[\![t]\!]$ in $\mathbf{FRing}$, which is the completion of $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[\![t]\!] \cong \mathcal{O}_X[\![t]\!]$. Therefore G is solid as well with $\mathcal{O}_G \cong \widehat{\mathcal{O}_X[\![t]\!]}$.

Moreover, if we further assume X is just a scheme, then

$$\mathcal{O}_G \cong \mathcal{O}_X \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}[\![t]\!] \cong \mathcal{O}_X[\![t]\!]$$

for now $\mathcal{O}_X$ is equipped with the discrete topology. A coordinate on G is the same as an isomorphism from G to $X \times \widehat{\mathbb{A}}^1$, which corresponds to a continuous map

$$u : \mathbb{Z}[\![t]\!] \to \mathcal{O}_G$$

which induces an isomorphism

$$\mathcal{O}_X[\![t]\!] \to \mathcal{O}_G.$$

Now since G is a group object, we have a map $G \times_X G \xrightarrow{\mu} G$, which corresponds to

$$\mathcal{O}_G \to \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G$$

of $\mathcal{O}_X$ modules. We also call the latter map μ, and it satisfies following properties.

Identity: There is a map $X \xrightarrow{e} G$ such that the composite

$$X \to G \to X$$

is identity. Moreover we require the composition

$$G \cong X \times_X G \xrightarrow{e \times id} G \times_X G \xrightarrow{\mu} G$$

equals the identity from G to itself.

Equivalently, there is a map $e : \mathcal{O}_G \to \mathcal{O}_X$, such that

$$\mathcal{O}_X \to \mathcal{O}_G \to \mathcal{O}_X$$

is identity and

$$\mathcal{O}_G \xrightarrow{\mu} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \xrightarrow{e \otimes id} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_G \cong \mathcal{O}_G$$

is identity.

Associativity:

$$\begin{CD} G \times_X G \times_X G @>{\mu \times id}>> G \times_X G \\ @V{id \times \mu}VV @VV{\mu}V \\ G \times_X G @>{\mu}>> G \end{CD} \qquad \begin{CD} \mathcal{O}_G @>{\mu}>> \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \\ @V{\mu}VV @VV{id \otimes \mu}V \\ \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G @>{\mu \otimes id}>> \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G. \end{CD}$$

Commutativity:

$$\begin{array}{ccc} G \times_X G & \xrightarrow{T} & G \times_X G \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & G & \end{array} \qquad \begin{array}{ccc} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G & \xrightarrow{T} & \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_G \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & \mathcal{O}_G & \end{array}$$

where T is transposition.

If we choose a coordinate on G, then we have an isomorphism from $\mathcal{O}_G$ to $\mathcal{O}_X[\![t]\!]$. The map μ now is

$$\mathcal{O}_X[\![t]\!] \longrightarrow \mathcal{O}_X[\![x, y]\!]$$

between $\mathcal{O}_X$ modules, which is determined by $f(x, y) = \mu(t) \in \mathcal{O}_X[\![x, y]\!]$. Such power series $f(x, y)$ is called a formal group law over $\mathcal{O}_X$, which satisfies

- $f(0, y) = y$;
- $f(x, f(y, z)) = f(f(x, y), z)$;
- $f(x, y) = f(y, x)$.

Remark 3. *The identity $\mathcal{O}_X[\![t]\!] \to \mathcal{O}_X$ can only be $t \mapsto 0$. Since this map is continuous between $\mathcal{O}_X$ modules, hence is determined by the image of t which is a nilpotent element n in $\mathcal{O}_X$. Note that*

$$f(x, y) = \sum f_i(x) y^i = \sum f_i(y) x^i$$

by commutativity. Hence we have

$$f(n,y) = \sum f_i(n)y^i = y$$

which implies that $f_0(n) = 0$*. Hence* $f_0(x)$ *is divided by* x^k *for some* k*. So does* $f_0(y)$*, that is*

$$f(x,y) = f_0(x) + f_0(y) + \cdots .$$

Hence k *must be 1 and* $n = 0$*.*

Example 2. *The additive formal group* $\mathbb{G}_a = \mathrm{Spf}(\mathbb{Z}[\![t]\!])$ *is a formal group over* $\mathbb{Z}$*. For any ring* R*,* $\mathbb{G}_a(R) \to \mathrm{Spec}(\mathbb{Z})(R)$ *is given by the inclusion of rings* $\mathrm{Nil}(R) \to *$*. The group structure on* $\mathbb{G}_a$ *is given by*

$$\mathrm{Nil}(R) \times \mathrm{Nil}(R) \to \mathrm{Nil}(R),\ (a,b) \mapsto a+b.$$

If we choose a coordinate $id : \mathbb{Z}[\![t]\!] \to \mathbb{Z}[\![t]\!]$*, the nwe get a formal group law* $f(x,y) = x + y$*.*

The multiplicative formal group $\mathbb{G}_m$ *over* $\mathbb{Z}$ *has the same underlying formal scheme. The group structure is given by*

$$\mathrm{Nil}(R) \times \mathrm{Nil}(R) \to \mathrm{Nil}(R),\ (a,b) \mapsto a+b+ab.$$

Use the same coordinate, we have a formal group law $f(x,y) = (1+x)(1+y) - 1$*.*

A morphism between two formal groups $\mathbb{G}$ over X and $\mathbb{H}$ over Y is just a commutative diagram in $\widehat{\mathfrak{X}}$ which respects the group structures of $\mathbb{G}$ and $\mathbb{H}$.

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{q} & \mathbb{H} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{H}} & \xrightarrow{q^*} & \mathcal{O}_{\mathbb{G}} \end{array}$$

Let x, y be coordinates on $\mathbb{G}$ and $\mathbb{H}$, then we have isomorphisms $\mathcal{O}_{\mathbb{G}} \cong \mathcal{O}_X[\![x]\!]$ and $\mathcal{O}_{\mathbb{H}} \cong \mathcal{O}_Y[\![y]\!]$ respectively. The morphism q^* sending y to a series $f(x) \in \mathcal{O}_X[\![x]\!]$, which satisfies

$$f(x +_{\mathbb{G}} x') = f(x) +_{\mathbb{H}} f(x').$$

Such series is called a homomorphism between formal group laws.

Example 3. *A crucial endomorphism from* $\mathbb{G}$ *to itself is multiplication by* p*. It is induced by*

$$[p] : \mathbb{G} \xrightarrow{\Delta} \underbrace{G \times_X \cdots \times_X G}_{p \text{ times}} \xrightarrow{\mu} G.$$

where Δ *is the diagonal. Choose a coordinate, we have* $[p](x) = x +_{\mathbb{G}} \cdots +_{\mathbb{G}} x$*,*

Suppose now X is over $\mathrm{Spec}(\mathbb{F}_p)$ and $q : \mathbb{G} \to \mathbb{H}$ over X with x, y are coordinates on them. Then there must be $a \neq 0 \in \mathcal{O}_X$ and r such that

$$q^*(y) = ax^r \bmod x^{r+1}.$$

Since q is a homomorphism, we have

$$a(x_0^r + x_1^r) = a(x_0 + x_1)^r \bmod (x_0, x_1)^{r+1}.$$

Let $r = p^n m$, we have

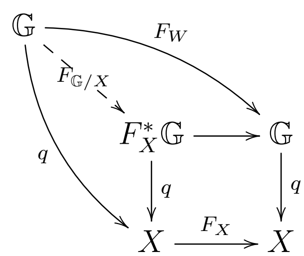
$$x_0^r + x_1^r = (x_0^{p^n} + x_1^{p^n})^m = x_0^r + m x_0^{r-p^n} x_1^{p^n} + \cdots \bmod (x_0, x_1)^{r+1}.$$

Hence m must be 1 and $r = p^n$ is a power of p.

Definition 7. *We call such n the strict height of q. We also let* Height(q) *to be the strict height of $\tilde{q} : \mathbb{G}_0 \to \mathbb{H}_0$ over the special fiber. Finally, we define* Height$(\mathbb{G})$ *to be* Height$([p] : \mathbb{G} \to \mathbb{G})$.

Remark 4. *Strict height is always not greater then height obviously. Moreover, we have $q^*(y) = g(x^{p^n})$. This is because $q^*(y) = f(x)$ must have no constant term due to the continuity. If $f'(0) \neq 0$, which means $f(x) = x + \cdots$, already meets the requirement. If $f'(0) = 0$, then the group law will force $f'(x) = 0$, which implies $f(x) = g(x^p)$.*

There is a geometric way to think of the strict height of a morphism $f : \mathbb{G} \to \mathbb{H}$ over X. Since X is over Spec$(\mathbb{F}_p)$, we have a Frobenius map $F_X : X \to X$. The pullback $F_X^*\mathbb{G}$ is also a formal group. If we choose a coordinate x on $\mathbb{G}$ and the induced coordinate y on $F_X^*\mathbb{G}$, then the formal group law on $F_X^*\mathbb{G}$ is given by $g^{(p)}(y, y')$, where g is the formal group law of $\mathbb{G}$ under the coordinate x and $g^{(p)}$ is the series obtained from replacing coefficients g_{ij} in g by g_{ij}^p.

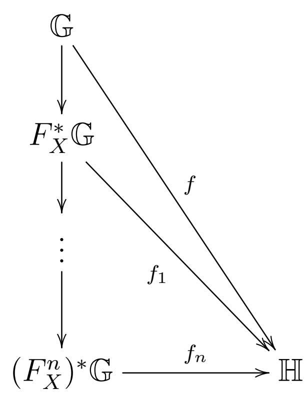


The commutativity of Frobenius maps induces a map $F_{\mathbb{G}/X} : \mathbb{G} \to F_X^*\mathbb{G}$, which is also a group homomorphism. Using the coordinates above, we have $F_{\mathbb{G}/X}^*(y) = x^p$.

Now suppose $f : \mathbb{G} \to \mathbb{H}$ is a group homomorphism with $f^*(y) = g(x^p)$ where x, y are coordinates on $\mathbb{G}$ and $\mathbb{H}$ respectively. From the expression of $f^*(y)$, we know that f factors through

$$\mathbb{G} \xrightarrow{F_{\mathbb{G}/X}} F_X^*\mathbb{G} \to \mathbb{H}.$$

The strict height of f corresponds to the height of the tower:



Proposition 2. *Let $f : \mathbb{G} \to \mathbb{H}$ be a nonzero homomorphism over X with* Height$(\mathbb{G})$ *finite. Then* Height$(\mathbb{G}) =$ Height$(\mathbb{H})$ *and* Height(f) *is finite.*

Proof. Just a direct computation. $\square$

3. Subschemes and Subgroups

Definition 8. *A map $f : X \to Y$ is a closed inclusion, if f is a regular monomorphism, i.e. it is the equalizer of $Y \to Y \coprod_X Y$ in $\widehat{\mathfrak{X}}$. A formal scheme X is a closed subscheme of Y if $f : X \to Y$ is a closed inclusion and X is a subfunctor of Y.*

Remark 5. *$Y \coprod_X Y$ is the pushout via $f : X \to Y$. In the category* **Top**, *a regular monomorphism is equivalent to an embedding.*

Example 4. *The map $f : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ induced by $f^* : A \to A/I$ is a closed subscheme. First $\mathrm{Spec}(A/I)(R)$ is the set of all maps from A to R which vanish on I, naturally a subset of $\mathrm{Spec}(A)(R)$. The equalizer of $\mathrm{Spec}(A) \rightrightarrows \mathrm{Spec}(A \times_{A/I} A)$ corresponds to the coequalizer of $A \times_{A/I} A \rightrightarrows A$, which is just A/I.*

Conversely, Suppose $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a closed inclusion. Then f is the equalizer of $\mathrm{Spec}(B) \rightrightarrows \mathrm{Spec}(C)$ which is equivalent to the spectrum of the coequalizer of $C \rightrightarrows B$, which is of the form B/I.

Example 5. *Suppose Y is a closed subscheme of X, i.e. $Y = \mathrm{Spec}(A/I)$ and $X = \mathrm{Spec}(A)$. The same argument implies that $X^\wedge_Y = \mathrm{Spf}(A^\wedge_I) \to X = \mathrm{Spec}(A)$ is a closed inclusion.*

Proposition 3. *If X is a formal scheme, Y is a scheme, then $f : X \to Y$ is a closed inclusion iff there are closed subschemes Y_j such that $X = \mathrm{colim}_j Y_j$.*

Proof. Suppose $X = \mathrm{colim}_j Y_j$, then from previous example, one has f is a closed inclusion.

Conversely, take a presentation $X = \mathrm{colim}_i X_i$, it is easy to verify that the canonical map $X_j \to X$ is a closed inclusion. Hence the composite $X_j \to Y$ is a closed inclusion, which implies X_j is a closed subscheme of Y. □

To investigate in general a closed inclusion $f : X \to Y$ between formal schemes, we need an understanding between categories $\widehat{\mathfrak{X}}_X$ and $\widehat{\mathfrak{X}}_{X_i}$.

Suppose $\{X_i\} : \mathcal{J} \to \widehat{\mathfrak{X}}$ is the filtered diagram with colimit X. Let $D_{\{X_i\}}$ be the category $\mathbf{Fun}(\mathcal{J}, \widehat{\mathfrak{X}})/\{X_i\}$ and $\widehat{\mathfrak{X}}_{\{X_i\}}$ be the full subcategory of $D_{\{X_i\}}$ with each such diagram a pullback for all arrows $u : i \to j$ in $\mathcal{J}$.

$$\begin{array}{ccc} Y_i & \xrightarrow{Y_u} & Y_j \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{X_u} & X_j \end{array}$$

Clearly, we have a functor $F : D_{\{X_i\}} \to \widehat{\mathfrak{X}}_X$ defined by taking colimit, and $G : \widehat{\mathfrak{X}}_X \to D_{\{X_i\}}$ via pullback.

Proposition 4. *The functor F is left adjoint to G, and the functor G is full and faithful. Moreover, G gives an equivalence between $\widehat{\mathfrak{X}}_X$ and $\widehat{\mathfrak{X}}_{\{X_i\}}$.*

Proof. A map from $F\{Y_i\}$ to Z over X is the same as a compatible system of maps $\{Y_i \to Z\}$ over X. Each $Y_i \to X$ factors through X_i, therefore such a map is equivalent to $Y_i \to X_i \times_X Z$ over X_i and the system of such maps is just a map $\{Y_i\} \to G(Z)$ in $D_{\{X_i\}}$, which implies the left adjointness.

Note that $FG(Y) = \operatorname{colim}_{\mathcal{J}} X_i \times_X Y$ and filtered colimits commutes with finite limits. Hence $\operatorname{colim}_{\mathcal{J}} X_i \times_X Y = X \times_X Y = Y$. The fully faithfulness of G follows, as

$$D_{\{X_i\}}(GX, GY) = \widehat{\mathfrak{X}}_X(FGX, Y) = \widehat{\mathfrak{X}}(X, Y).$$

The equivalence of categories follows from intuition but requires some work. □

According to this proposition, suppose $F : X \to Y$ is a closed inclusion, with Y solid. Then f is equivalent to an element in $\widehat{\mathfrak{X}}_{\{Y_i\}}$ with each $X_i \to Y_i$ a closed inclusion, which yields that each X_i is solid. Hence $X = \operatorname{colim} X_i$ is again solid. This proves the following conclusion.

Proposition 5. *A closed subscheme of a solid formal scheme is again solid.*

Proof. This follows from proposition 4. Moreover, suppose X is a solid formal subscheme of solid formal subscheme Y, then we have $\mathcal{O}_X = \mathcal{O}_Y/J$ for some ideal J. □

Definition 9. *Let C be a formal curve over X, i.e. $C \cong X \times \widehat{\mathbb{A}}^1$ and D is a closed subscheme of C. Suppose X is a scheme, we say D is a divisor of degree n if D is also a scheme and $\mathcal{O}_D$ a free module of rank n over $\mathcal{O}_X$. For general X, we say D is a divisor if for all scheme X', $D \times_X X'$ is a divisor of $C \times_X X'$.*

Now suppose X is a scheme with $X = \operatorname{Spec}(R)$. Choose a coordinate x on C, we have $C \cong \operatorname{Spf}(R[\![x]\!])$. Since D is a divisor of C, we have $D = \operatorname{Spec}(R[\![x]\!]/J)$ for some ideal J such that $x^k \in J$ for some k. Let $\lambda(x)$ be the R-endomorphism of $\mathcal{O}_D = R[\![x]\!]/J$ given by multiplying x and let $f_D(t)$ denote its characteristic polynomial.

Suppose D is of degree n, then $f_D(t)$ is a degree n monic polynomial. By Cayley-Hamilton, $f_D(x) = 0 \in \mathcal{O}_D$, which implies $f_D(x) \in J$. While $R[\![x]\!]/f_D(x)$ is also a free module of rank n over R and $\mathcal{O}_D$ is a quotient of it, which is also free of rank n. Hence $\mathcal{O}_D = R[\![x]\!]/f_D(x)$.

Moreover, since $x^k \in J$, if R is a field, we have $f_D(t) = t^n$. If R is not a field, by passing to the fraction field of $R/(p)$, we have all coefficients of $f_D(t) = \sum_{i=1}^n a_i x^i$ lies in (p) but $a_n = 1$. Hence they actually lie in $\operatorname{Nil}(R)$.

Proposition 6. *There is a formal scheme $\operatorname{Div}_n^+(C)$ over X, which classifies all effective divisors of degree n on Y over X. Moreover, given a coordinate on C, $\operatorname{Div}_n^+(C) \cong X \times \widehat{\mathbb{A}}^n$.*

Suppose $n = 1$, a divisor D here is just a section of C over X. Conversely, a section is of course a closed subscheme of C which is finite and flat over X. Hence we have

$$\operatorname{Div}_1^+(C) = C$$

over X. The universal divisor D^u over $\operatorname{Div}_1^+(C) = C$ is a closed subscheme of $C \times_X C$. Given a coordinate X on C, the polynomial $f_{D^u}(t) = t - x$.

If we have two divisors D and D' over X, we define $D + D'$ to be the divisor corresponds to $f_D(t) f_{D'}(t)$. This defines a map

$$\operatorname{Div}_m^+(C) \times_X \operatorname{Div}_n^+(C) \to \operatorname{Div}_{m+n}^+(C),$$

which corresponds to

$$\mathcal{O}_X[\![x_0, x_1]\!] \to \mathcal{O}_X[\![x, y]\!]$$
$$x_0 \mapsto xy$$
$$x_1 \mapsto x + y$$

when m, n equals 1.

Hence we have a map $C_X^n/\Sigma_n \to \mathrm{Div}_n^+(C)$ for the commutativity of addition of divisors, which is an isomorphism.

Proposition 7. *The map $C_X^n/\Sigma_n \to \mathrm{Div}_n^+(C)$ is an isomorphism, and the universal divisor of degree n has the polynimial*

$$f_D(t) = \prod_k (x - a_k).$$

Proof. We work in the opposite category. $C_X^n/\Sigma_n \cong \mathrm{Spf}(\mathcal{O}_X[\![\sigma_1, \cdots, \sigma_n]\!])$ where σ_i is the $i'th$ elementary symmetric polynomial. The map $C_X^n \to \mathrm{Div}_n^+(C)$ induces

$$\mathcal{O}_{\mathrm{Div}_n^+(C)} = \mathcal{O}_X[\![a_1, \cdots, a_n]\!] \to \mathcal{O}_X[\![x_1, \cdots, x_n]\!]$$

with a_i sending to σ_i, the $i'th$ elementary polynomial. Therefore $\mathrm{Div}_n^+(C)$ is equal to C_X^n/Σ_n. □

Definition 10. *A subgroup of G is a divisor of G which is also a subgroup.*

Proposition 8. *Suppose K is a subgroup of G, then degree K is a power of p.*

Proof. Since K is a subgroup, we know that $\mathcal{O}_K \cong \mathrm{Spf}(\mathcal{O}_X[\![x]\!]/f_K(x))$. We also have

$$\begin{array}{ccc} K \times_X K & \longrightarrow & G \times_X G \\ \downarrow & & \downarrow \mu \\ K & \longrightarrow & G. \end{array}$$

The multiplication of K must factor through K. In another word,

$$\begin{array}{ccc} \mathcal{O}_X[\![t]\!] & \longrightarrow & \mathcal{O}_X[\![t]\!]/f_K(t) \\ \downarrow \mu^* & & \downarrow \\ \mathcal{O}_X[\![x, y]\!] & \longrightarrow & \mathcal{O}_X[\![x, y]\!]/(f_K(x), f_K(y)). \end{array}$$

This means

$$f_K(g(x, y)) = 0 \bmod f_K(x), f_K(y).$$

Now checking the coefficients like we already done after Example 3, the degree of f_K must be a power of p. □

Remark 6. *The group structure also requires that the identity lies in K, this is the same as*

$$X \to G$$

factors through

$$X \dashrightarrow K \longrightarrow G.$$

This is equivalent to require that $f_K(x) \in (x)$. Hence if K is a subgroup of G, we must have f_K is a polynomial of degree p^r and divided by x, i.e. it has no constant terms.

Proposition 9. *Suppose K is a subgroup of G with degree p^m, then $[p]_K^m = 0$. Hence $K \leq G(m) = \ker([p]_G^m)$.*

Proof. For any R, $K(R)$ is a subgroup of $G(R) = \mathrm{Nil}(R)$, with elements satisfying the relation $f_K = 0$. All solutions of f_K in R are automatically nilpotent for all coefficients of f_K are nilpotent. Suppose there are some solutions, say α, not in R. We can embed $\mathrm{Nil}(R)$ into $\mathrm{Nil}(R[\alpha])$. So we can assume f_K has all solutions in R. Hence $K(R)$ is a group of order p^m. Thus any elements has order p^m. In another words

$$G(R) \xrightarrow{[p]^m} G(R),\ x \mapsto [p]^m(x) = x +_G \cdots +_G x$$

restricts to

$$K(R) \xrightarrow{[p]^m} K(R),\ x \mapsto [p]^m(x) = 0.$$

□

Having define subgroups of a formal group, we then consider the quotient groups. As in group theory, we define G/K to be the coequalizer

$$G \times_X K \underset{\pi}{\overset{\mu}{\rightrightarrows}} G \longrightarrow G/K.$$

On the level of functions, we have

$$\mathcal{O}_{G/K} \longrightarrow \mathcal{O}_G \underset{\pi^*}{\overset{\mu^*}{\rightrightarrows}} \mathcal{O}_G \otimes_{\mathcal{O}_X} \mathcal{O}_K.$$

Let x be a coordinate on G, and y be $N_\pi \mu^* x \in \mathcal{O}_G$.

Theorem. *Let K be a subgroup of G with degree p^m. The element y actually lies in $\mathcal{O}_{G/K}$, which satisfies*

(1) $y = x^{p^m} \mod \mathfrak{m}_X$.
(2) $\mathcal{O}_{G/K} = \mathcal{O}_X[\![y]\!]$
(3) *G/K has a natural structure as a formal group.*

Proof. First notice that there is an automorphism θ from $G \times_X K$ to it self, which sending (a, b) to $(a - b, b)$. Therefore $\pi = \mu\theta$, and π finite flat implies μ finite flat. Now we have two pullback diagrams with π' forget the third component.

$$\begin{array}{ccc} G \times_X K \times_X K & \xrightarrow{\mu \times 1} & G \times_X K \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times_X K & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccc} G \times_X K \times_X K & \xrightarrow{1 \times \mu} & G \times_X K \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times_X K & \xrightarrow{\pi} & G \end{array}$$

The second one is a pullback for there is a unique automorphism extending the following diagram

$$\begin{array}{ccc} G \times_X K \times_X K & & \\ \downarrow & \searrow^{1 \times \mu} & \\ G \times_X K \times_X K & \xrightarrow{\pi \times 1} & G \times_X K \end{array}$$

which sends (a, b, c) to $(a, b, b + c)$.

The maps involved in above diagrams are all finite flat maps, therefore we have

$$\mu^* N_\pi = N_{\pi'}(\mu \times 1)^* \quad , \quad \pi^* N_\pi = N_{\pi'}(1 \times \mu)^*.$$

Follow this and $\mu(1 \times \mu) = \mu(\mu \times 1)$ yields $y \in \mathcal{O}_{G/K}$.

For (a), let $j : K \to G$ be the inclusion, and $i : K \to G \times_X K$ with $i(a) = (0, a)$. Hence $j = \mu i$ and $\pi i = 0$. We have the following.

$$\begin{array}{ccccc} K & \xrightarrow{i} & G \times_X K & \xrightarrow{\mu} & G \\ \downarrow & & \downarrow \pi & & \\ X & \xrightarrow{0} & G & & \end{array}$$

Now $j^*y = i^*\mu^*y = i^*\pi^*y = 0^*y = 0$. This implies that y is divisible by f_K.

Recall that y is the norm of μ^*x under π. After mod $\mathfrak{m}_X$, $f_K(z)$ becomes z^{p^m} and we can write μ^*x as

$$x + a_1(x)z + a_2(x)z^2 + \cdots + a_{p^m-1}(x)z^{p^m-1} \in k[\![x]\!][z]/z^{p^m}$$

where k is the residue field of $\mathcal{O}_X$.
A direct computation implies

$$\mu^*x(1, z, \cdots, z^{p^m-1}) = (1, z, \cdots, z^{p^m-1}) \begin{bmatrix} x & & & \\ a_1(x) & x & & \\ \vdots & \vdots & \ddots & \\ a_{p^m-1}(x) & a_{p^m-2}(x) & \cdots & x \end{bmatrix}.$$

Therefore $y = N_\pi \mu^*x = x^{p^m}$ mod $\mathfrak{m}_X$. This completes (a). Moreover y is a unit multiple of $f_K(x)$.

For part (b), suppose $u \in \mathcal{O}_{G/K}$. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{G/K} & \longrightarrow & \mathcal{O}_G \\ 0^* \downarrow & & \downarrow j^* \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_K \end{array}$$

The vertical map 0^* is just taking constant terms. From this we know $j^*(u - u(0)) = 0$. Hence $u - u(0)$ is divided by y. We can write $u = u(0) + u'y$ with $u' \in \mathcal{O}_G$. Since $u'y \in \mathcal{O}_{G/K}$, we conclude that $\pi^*(u'y) = \pi^*(u')\pi^*(y) = \mu^*(u'y) = \mu^*(u')\mu^*(y)$. The element $\pi^*(y) = \mu^*(y)$ is not a zero divisor, which implies $u' \in \mathcal{O}_{G/K}$. By induction, we have

$$\mathcal{O}_{G/K} = \mathcal{O}[\![y]\!].$$

Part (c) is obvious. G/K has the induced formal group law from G, with the coordinate y induced from x. And the projection $G \to G/K$ is the category cokernel of $K \to G$. □

4. Level Structures

Suppose there are Y and a formal group G over X, then $\Gamma(Y, G)$, the set of all maps from Y to G over X forms a group via

$$Y \xrightarrow{(f,g)} G \times_X G \xrightarrow{\mu} G.$$

In terms of functions, let x be a coordinate on G and $a, b \in \Gamma(Y, G)$ corresponds to $\mathcal{O}_X[\![x]\!] \to \mathcal{O}_Y$, sending x to a, b in $\mathcal{O}_Y$ respectively, then the composite ab is given by

$$\begin{aligned} \mathcal{O}_X[\![x]\!] &\to \mathcal{O}_X[\![x, y]\!] \to \mathcal{O}_Y \\ x &\mapsto \mu(x, y) \mapsto \mu(a, b), \end{aligned}$$

where $\mu(x, y)$ is the corresponding formal group law over G associated to the coordinate x.

One can check the identity element in $\Gamma(Y,G)$ is

$$Y \to X \xrightarrow{0} G,$$

where 0 is the identity element of G.

To analyze this group, we can use other groups map to it, in particular, finite abelian $p-$groups. This suggests us to consider the functor

$$Y \to \operatorname{Hom}(A, \Gamma(Y,G))$$

from (formal)schemes to sets.

Suppose $A = \mathbb{Z}/p^k$. A group homomorphism ϕ from A to $\Gamma(Y,G)$ is determined by $s = \phi(1) \in \Gamma(Y,G)$, which satisfies $[p^k]_G(s) = 0$. In terms of coordinates, let s denote the map $\mathcal{O}_X[\![x]\!] \to \mathcal{O}_Y$, sending x to $s \in \mathcal{O}_Y$. Then $[p^k]_G(s) = 0$ requires the composite

$$Y \xrightarrow{s} G \xrightarrow{\Delta} G \times_X G \times_X \cdots \times_X G \xrightarrow{\mu} G$$

is identity in $\Gamma(Y,G)$. In another words, the map

$$\mathcal{O}_X[\![x]\!] \to \mathcal{O}_X[\![x]\!] \to \mathcal{O}_Y$$
$$x \mapsto [p^k]_G(x) \mapsto [p^k]_G(s)$$

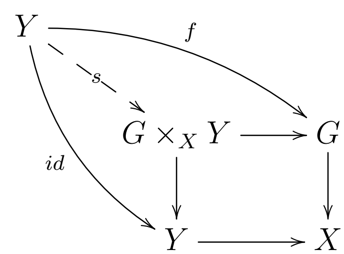
should be zero. Therefore in this case, this functor is represented by the subgroup scheme

$$G(k) := \ker([p^k]_G : G \to G).$$

In general, let $A = \prod_{k=0}^{r-1} \mathbb{Z}/p^{d_k}$. This functor, denoted by $\operatorname{Hom}(A,G)$, is also representable. We have $\operatorname{Hom}(A,G) = \prod_{k=0}^{r-1} G(d_k)$ over X with

$$\mathcal{O}_{\operatorname{Hom}(A,G)} = \mathcal{O}_X[\![x_0, \cdots, x_{r-1}]\!]/([p^{d_0}]_G(x_0), \cdots, [p^{d_{r-1}}]_G(x_{r-1})).$$

This is a finite free module over $\mathcal{O}_X$, hence $\operatorname{Hom}(A,G)$ is finite flat over X. If the height of G is n, then the degree of $\operatorname{Hom}(A,G)$ over X is $|A|^n$.



Given a map $f : Y \to G$ over X, we have a section s of G over Y and therefore a divisor $[s]$ of G over Y with degree 1. We define

$$[\phi A] := \sum_{a \in A} [\phi(a)]$$

which is a divisor of degree $|A|$. We put $A(k) := \ker(p^k : A \to A)$. We also put $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, hence $\Lambda(m) = (\mathbb{Z}/p^m)^n$.

Definition 11. *A level-A structure on G over an X-scheme Y is a map $\phi : A \to \Gamma(Y,G)$, such that $[\phi A(1)] \leq G(1)$ as divisors. A level-m structure means a level-$\Lambda(m)$ structure.*