Dyer-Lashof Theory and Algebras

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References I

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Contents

A group is a set G with $c : G \times G \to G$ and $i : G \to G$.

Slogan: An algebraic structure is equivalent to some commutative diagrams.

Algebraic Theories

Definition 1.1

An ***algebraic theory*** *T is a category with objects $\{T^0, T^1, \dots\}$. And there are maps $\pi_i : T^n \to T^1$ for all $n \geq 0, 1 \leq i \leq n$, such that $T(T^k, T^n) \xrightarrow{\pi_i} \prod_{i=1}^{n} T(T^k, T^1)$ is a bijection.*

This means T^n is isomorphic to n-fold product of T^1.

Definition 1.2

A ***model*** *for an algebraic theory T is a functor $F : T \to Sets$.*

Question: What theory T would stand for the theory of groups?

$T^1 = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, or something else? $T^n =$?

$T^n = \langle x_1, \ldots, x_n \rangle$ the free object of n generators in Grps.
$T(T^n, T^1) = Grps(\langle x_1 \rangle, \langle x_1, \ldots, x_n \rangle)$.

$$x_1 \mapsto x_1 x_2 \in T(T^2, T^1)$$
$$x_1 \mapsto x_1^{-1} \in T(T^1, T^1)$$

represent the structure maps required for a group.

Example: F = full subcategory of Alg_R, with obejects $\{F_0, F_1, \ldots\}$, $F_i = R[x_1, \ldots, x_i]$. Let $T = F^{op}$. Let π_i be the following:

$$T^n \xrightarrow{\pi_i} T^1$$
$$R[x_1, \ldots, x_n] \leftarrow R[x_1]$$
$$x_i \leftarrow x_1.$$

Then T is an algebraic theory of commutative R algebras, denoted by C_R and any model $A : T \to \mathit{Sets}$ gives $A(T^1)$ a structure of R algebras.

Multiplication: $x_1 \mapsto x_1 x_2$.
Addition: $x_1 \mapsto x_1 + x_2$.

Free Models

Given an algebraic theory T, and a model A, we usually abbreviate the notation $A(T^1)$ by A.

$F_T(n) = T(T^n, -)$: the ***free model*** of n generators.

For example, in the algebraic theory of commutative R-algebras:

$$F_T(n)(T^1) = T(T^n, T^1) = Alg_R(R[x], R[x_1, \dots, x_n]) \cong R[x_1, \dots, x_n].$$

Morphisms of Theories

Let $\phi : T \to H$ be a functor between two theories, such that $\phi(T^k) = H^k$ with projection maps sent to projection maps.

$$\phi^* : Model_H \to Model_T$$

Example: the theory of abelian groups $\to$ the theory of Rings.

T is a ***COT***, if $\exists\, \phi : C_R \to T$ for some commutative ring R.

Graded Algebraic Theories

Let C be a fixed set, and $\mathbb{N}[C]$ be the set generated by C.

Definition 1.3

*A C-**graded theory** T is a category with objects $\{T^d\}_{d\in N[C]}$, together with, for each $d = \Sigma_{c\in C} d_c[c] \in \mathbb{N}[C]$, a specified identification of T^d with the product $\prod (T^{[c]})^{\times d_c}$.*

Example: Let C be $\mathbb{N}$, we can define the theory of graded R algebras as before. Let $F_{[c]} = R[x_c]$, where x_c has degree c, and $T^{[c]} = F_{[c]}^{op}$.

Addition: $T^{2[c]} \to T^{[c]}$, $x_c \mapsto \alpha_c + \beta_c$.
Multiplication: $T^{[c]+[c']} \to T^{[c+c']}$, $x_{c+c'} \mapsto x_c \cdot x_{c'}$.

Dyer-Lashof Theory

We want an algebraic theory which describes the algebraic structure for $\pi_*(A)$, where A is an R-algebra.
$R =$ commutative S-algebra.
$M =$ an R-module. Note: $[R, M]_R \cong [S, M]_S \cong \pi_0 M$.
Free commutative $R-$algebra on M:

$$\mathbb{P}_R(M) = \bigvee_{m \geq 0} \mathbb{P}_R^m(M) \cong \bigvee_{m \geq 0} M \wedge_R \cdots \wedge_R M/\Sigma_m.$$

Given a commutative S-algebra R, let DL_R denote the $\mathbb{Z}$ graded theory T defined by

Definition 1.4

$$T(T^{[c_1]+\cdots+[c_m]}, T^{[d_1]+\cdots+[d_n]}) \cong hAlg_R(\mathbb{P}_R(R \wedge (S^{d_1} \vee \cdots \vee S^{d_n})), \mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee S^{c_m}))).$$

Remark: The $R-$mod spectrum $R \wedge (S^{c_1} \vee \cdots \vee S^{c_m})$ can be viewed as R-module $R\{x_{c_1}, \ldots, x_{c_m}\}$ in commutative algebra. And $\mathbb{P}_R$ turns the R-module into R-algebra $R[x_{c_1}, \ldots, x_{c_m}]$.

We see that taking homotopy groups defines a functor

$$\pi_* : hAlg_R \to Model_{DL_R}.$$

For we have

$$\pi_q(A) \cong hMod_S(S^q, A) \cong hMod_R(R \wedge S^q, A) \cong hAlg_R(\mathbb{P}_R(R \wedge S^q), A).$$

Thus, DL_R describes all homotopy operations on commutative R-algebras.[Rezk]

If A is a commutative S-algebra, then $R \wedge_S A$ is a commutative R-algebra. Hence we have the composite:

$$R_* : A \mapsto R \wedge_S A \xrightarrow{\pi_*} R_* A$$

Thus DL_R describes homology operations CAlg_S.
If T is a space, then there is a commutative R-algebra

$$R^T \stackrel{def}{=} Hom(\Sigma_+^\infty T, R).$$

Thus we have the functor $R^* : T^{op} \to Model_{DL_R}$, the R cohomology of a space is a DL_R model.

Operations

Let $f \in F_T(n)(T^1)$, and $a_1, \ldots, a_n \in A(T^1)$, where A is any model of T.

Let $f \propto (a_1, \ldots, a_n)$ denote the image of f under the map $F_T(n) \to A$ sending x_i to a_i. We call the function:

$$f \propto : A^n \to A$$

the ***operation*** associated to f.

Example: Let $T = C_R$, then we know that

$$F_T(n)(T^1) = Alg_R(R[x], R[x_1, \ldots, x_n]) \cong R[x_1, \ldots, x_n].$$

We abbreviate $F_T(n)(T^1)$ by $R\{x_1, \ldots, x_n\}$.

Hence $f \propto (a_1, \ldots, a_n)$ is just $f(a_1, \ldots, a_n)$.

If T is a COT, then we have

$$F_T(n) \cong F_T(1) \otimes_R \cdots \otimes_R F_T(1).$$

Hence we may focus on operations in $F_T(1)$.

They satisfies:

$$
\begin{aligned}
x \propto a &= a \\
(f \propto g) \propto a &= f \propto (g \propto a) \\
(f + g) \propto a &= f \propto a + g \propto a \\
(fg) \propto a &= (f \propto a)(g \propto a) \\
r \propto a &= r
\end{aligned}
$$

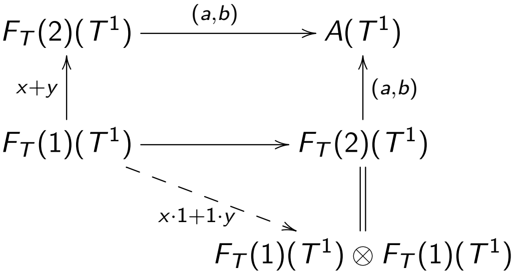
Additive Operations

Furthermore, if $f \in R\{x\}$ satisfies

$$f \propto (a_1 + a_2) = f \propto a_1 + f \propto a_2$$

We say it is an ***additive operation***, denoted the set of all additive operations by $\mathcal{A}$.

$\mathcal{A}$ is an associative ring with product $\propto$, but not commutative in general.



$R\{x\}$ has a additive coproduct $\Delta : R\{x\} \to R\{x_1, x_2\}$ given by $x \mapsto x_1 + x_2$, corresponds to the structure map under addition.

Additive operations are those elements with ***primitive*** image.

Examples

C_R: $\mathcal{A} = R \cdot x \cong R$ when R torsion free. If R is a field of char=p, then $\mathcal{A} \cong R\langle \phi \rangle$, where ϕ is Frobenius and

$$\phi r = r^p \phi.$$

Let T be the theory of R-algebras with G-action.
T is a COT. $R\{x\} = R[x^g : g \in G]$, $\mathcal{A} \cong R[G]$

$$\mathbb{P}^m(S^0) = E\Sigma_m \times_{\Sigma_m} (S^0)^{\wedge m} / E\Sigma_m \times_{\Sigma_m} * = B\Sigma_m^+.$$

$$\begin{aligned}\mathbb{P}^m(S^d) &= E\Sigma_m \times_{\Sigma_m} (S^d)^{\wedge m} / E\Sigma_m \times_{\Sigma_m} * \\ &= B\Sigma_m \times dV_m/\text{boundary} = B\Sigma_m^{dV_m}.\end{aligned}$$

$H\mathbb{F}_2$

$F_T([c_1] + \cdots [c_m]) = \mathbb{P}_H(H \wedge (S^{c_1} \vee \cdots S^{c_m}))^{op}$ and

$$\begin{aligned} F_T([c_1] + \cdots [c_m])_{[d]} &= hAlg_H(\mathbb{P}_H(H \wedge S^d), \mathbb{P}_H(H \wedge (S^{c_1} \vee \cdots S^{c_m}))) \\ &= \pi_d(H \wedge \mathbb{P}S^{c_1} \wedge \cdots \mathbb{P}S^{c_m}) \\ &= H_d(\mathbb{P}S^{c_1} \wedge \cdots \mathbb{P}S^{c_m}) \end{aligned}$$

Hence by Künneth formula, we have

$$F_T([c_1] + \cdots [c_m])_* = F_T([c_1])_* \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} F_T([c_m])_*$$

Hence $\mathrm{DL}_{H\mathbb{F}_2}$ is a COT.

Now consider operations $\pi_c A \to \pi_{c+s} A$.

$$hAlg_H(\mathbb{P}_H(H \wedge S^{c+s}), \mathbb{P}_H(H \wedge S^c)) = \pi_{c+s}\mathbb{P}_H(H \wedge S^c)$$

Restrict our attention to $\pi_{c+s}\mathbb{P}^2_H(H \wedge S^c)$. Now

$$\pi_{c+s}\mathbb{P}^2_H(H \wedge S^c) \cong \pi_{s+c}(H \wedge \mathbb{P}^2 S^c) \cong H_{s+c}(B\Sigma_2^{cV_2}) \cong H_{s-c}(\mathbb{R}P^\infty).$$

Hence we have

$$\pi_{c+s}\mathbb{P}^2_H(H \wedge S^c) = \begin{cases} 0 & \text{if } s < c \\ \mathbb{F}_2 & \text{if } s \geq c \end{cases}$$

So we have obtained $Q^s : \pi_c A \to \pi_{c+s} A$ for any $c \leq s$.
Extended Q^s over those $s < c$ by setting $Q^s = 0$.

There for a model for $DL_{H\mathbb{F}_2}$ is at least a graded $\mathbb{F}_2$ algebra equipped with such $Q^s : A_c \to A_{c+s}$.

These Q^s satisfies some relations.

Dyer-Lashof Algebras for $H\mathbb{F}_2$

For $H\mathbb{F}_2$ algebra A, $\pi_* A$ is a graded algebra equipped with Q^s

1. Q^s are additive,
2. $Q^s(a) = 0$ for $s < |a|$,
3. $Q^s(a) = a^2$ for $s = |a|$,
4. Cartan formula

$$A^s(ab) = \Sigma_{i+j=s} Q^i(a) Q^j(b),$$

5. Adem relations

$$Q^r Q^s = \Sigma_{i+j=r+s} \binom{j-s-1}{2j-r} Q^i Q^j$$

for $r > 2s$.

K Theory

We focus on $\pi_0 A$ for a K-algebra A.

$$hAlg_k(\mathbb{P}_K(K), \mathbb{P}_K(K)) = \pi_0(K \wedge \mathbb{P}S^0) = K_0(\bigvee_m B\Sigma_m).$$

The crucial thing is to compute $K_0 B\Sigma_m$.

$$K^i(B\Sigma_m) = \begin{cases} R(\Sigma_m)^\wedge_I & \text{i even} \\ 0 & \text{i odd} \end{cases}$$

Mod p K-Theory

In mod p cases, things are getting easier.

$$K^i(B\Sigma_m;\mathbb{Z}/p) = \begin{cases} R(\Sigma_m)_I^\wedge \otimes \mathbb{Z}/p & \text{i even} \\ 0 & \text{i odd} \end{cases}$$

and universal coefficient theorem:

$$K^i(B\Sigma_m;\mathbb{Z}/p) = Hom(K_i(B\Sigma_m;\mathbb{Z}/p),\mathbb{Z}/p)).$$

Total Power Operation and Individual Operations

Suppose A is an R-algebra, for $f : R \to A$, define $P_m(f)$ to be:

$$R \wedge B\Sigma_m^+ \cong \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f)} \mathbb{P}_R^m(A) \hookrightarrow A.$$

That is $P_m : \pi_0 A \to \pi_0(A^{B\Sigma_m^+})$.

Precomposing $\alpha \in \pi_0(R \wedge B\Sigma_m^+) \cong R_0(B\Sigma_m)$ gives individual operation Q^α :

$$R \xrightarrow{\alpha} R \wedge B\Sigma_m^+ \cong \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f)} \mathbb{P}_R^m(A) \hookrightarrow A.$$

Let $A = K^X$, then

$$\begin{aligned}\pi_0(A^{B\Sigma_m^+}) &= \pi_0(K^X)^{B\Sigma_m^+} \\ &= Mod_S(X \wedge B\Sigma_m^+, K) \\ &= K_{\Sigma_m}(X)\end{aligned}$$

$$P_m : K(X) \to K_{\Sigma_m}(X^{\times m}) \xrightarrow{\delta^*} K_{\Sigma_m}(X) \cong K(X) \otimes_{\mathbb{Z}} R(\Sigma_m).$$

Then any $u \in Hom(R(\Sigma_m), \mathbb{Z})$ will give an operation on $K(X)$, for example the Adams operations ψ^m.

Thank You!