硕士学位论文

代数拓扑中的计算方法

COMPUTATIONAL METHODS IN ALGEBRAIC TOPOLOGY

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摘 要

这篇硕士论文启发自拓扑空间之间的连续映射在同伦意义下的分类问题，由此引出针对这种问题的计算方法的探究。着重介绍了两种重要尽管繁琐的计算工具，分别是谱序列和上同调运算，并运用这些工具对一些例子进行了计算。论文的第三章主要介绍了谱序列的构造和基本性质，研究了一种特别的谱序列——塞尔同调及上同调谱序列。运用塞尔上同调谱序列重新计算了三维球面的四维同伦群，之后通过引入塞尔类重新证明了除 0 维外所有球面稳定同伦群都是有限交换群。第四章的前半部分介绍了上同调运算，特别是斯廷罗德构造的斯廷罗德代数。它将同调函子从拓扑空间范畴到环范畴推广为到斯廷罗德代数上的模范畴，而且保留了之前的环结构。论文运用斯廷罗德代数重新计算了球面上正交切向量场的最大个数的一个上界。另一个斯廷罗德代数的应用是亚当斯谱序列，用于计算两个拓扑空间之间的由映射的稳定同伦类构成的群，以球面二维稳定同伦群的 2 分量的实际计算作为应用。第四章的后半部分独立于前半部分介绍了一种广义上同调理论——K 理论，以及它上面附带的上同调运算。运用它们可以得到代数拓扑中关于霍普夫不变量为 1 的经典定理以及实线性空间作为实数域上的可除代数的结论。

关键词：塞尔谱序列；亚当斯谱序列；上同调运算；稳定同伦群；K 理论

ABSTRACT

This thesis is motivated by a classification problem of continuous maps between topological spaces up to homotopy. For this purpose, it then turns to some computational methods that are useful albeit complicated, namely, spectral sequences and cohomology operations, and uses them to calculate with some interesting examples. Chapter 3 introduces the general construction and properties of a spectral sequence, and specializes these ideas to the Serre spectral sequences for homology and cohomology with explicit constructions. We calculate $\pi_4(S^3)$ and reprove the finiteness of stable homotopy groups in positive dimensions through an argument with the Serre classes. Chapter 4 introduces a class of important operations on cohomology, carrying a lot of extra information and structures than classical cohomology theories. They form the Steenrod algebras $\mathcal{A}_p$ for p a prime number. Their applications include giving an upper bound for orthogonal tangent vector fields on spheres and Adams spectral sequences. The latter becomes an extremely powerful tool in algebraic topology, calculating the stable homotopy classes of maps between two topological spaces. As an example, we calculate the 2-component of π_2^s. The last two sections of Chapter 4 introduce K-theory as a generalized cohomology theory and the Adams operations as its cohomology operations. Using them we deduce two classical facts in algebraic topology, namely, Adams's theorem on Hopf invariants and the classification of finite-dimensional division algebras over $\mathbb{R}$.

Keywords: Serre spectral sequence; Adams spectral sequence; cohomology operation; stable homotopy group; K-theory

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CHAPTER 1 INTRODUCTION

Algebraic topology is a branch of mathematics that uses tools from abstract algebra to study topological spaces and continuous maps between them. The ultimate goal is to classify all topological spaces up to homeomorphism using algebraic invariants, though usually most classify up to homotopy equivalence. So the computation methods are significant parts in algebraic topology.

The first concerned invariant of a space X, in some sense, is the fundamental group $\pi_1(X)$, which roughly speaking is sort of loops in the space X. Two loops are viewed as the same if one can deform continuously to the other. After choosing a basepoint x_0 in X, we can equip the set of loops starting and ending at x_0 a group structure, denoted by $\pi_1(X, x_0)$. Henri Poincaré defined the fundamental group in 1895 in his paper ”Analysis situs”. The concept emerged in the theory of Riemann surfaces, in the work of Bernhard Riemann, Poincaré, and Felix Klein. It describes the monodromy properties of complex-valued functions, as well as providing a complete topological classification of closed surfaces. Even it is the simplest invariant in algebraic topology, the computation of fundamental groups is not easy. Van Kampen's theorem and covering space theory are basic tools for computing it. Former allow us to decompose the fundamental group of X into simpler spaces whose fundamental groups are easier to obtain. The latter gives a correspondence of all subgroups of $\pi_1(X)$ and all covering spaces of X, which is surprisingly analog of Galois correspondence.

The simplest invariant means it captures the least information of spaces. Fundamental groups only capture little information. For instance, S^2 and S^3 both have trivial fundamental groups, but they are not homeomorphic. A more powerful tool called homology was introduced by Henri Poincaré. The main idea is to count the ”holes” in a space X. It is more complicated than fundamental groups but fortunately we have more tools to compute them. For example, simplicial homology, cellular homology. After that, we have a nice version called singular homology which has its own advantage in theoretical aspect. And these homology theories are coherent in most situations. Homology groups are abelian, which makes one get comfortable comparing with nonabelian groups such as fundamental groups. It can also detect higher dimensional information, we can use homology to distinguish S^2 from S^3 in the previous example. In fact, we can use it to distinguish manifolds in different dimensions. Homology has many applications, such as Brouwer fixed point theorem in all dimensions, degree of a map, hairy-ball theorem and so on. But it is not enough, $S^1 \vee S^1 \vee S^2$ and T^2 can not be distinguished by homology theories, and they are not even homotopy equivalent.

Actually, there may not exist a perfect theory to detect all different spaces, at least for now. To improve homology theories, algebraic topologists started to observe functions from chain groups to a given group G. This leads to the dual conception of homology theories, the cohomology theories. They are some abelian groups such as homology groups, but there are more structures on cohomology theories, for instance, the cup product, which endows a ring structure on a cohomology theory. One important structure is cohomology operations, especially Steenrod squares and Steenrod powers, which gives a cohomology theory $\mathcal{A}$-module structure after the ring structure. Using these structures, we can detect more information and classify spaces more accurately. We can never find that $S^2 \vee S^4$ is not homotopic equivalent to $\mathbb{CP}^2$ just using homology or additive structure of cohomology. But when we look at the ring structure of cohomology, the power of each element in $\mathrm{H}^*(S^2 \vee S^4; \mathbb{Z})$ is zero, while in $\mathrm{H}^*(\mathbb{CP}^2; \mathbb{Z})$, the generator has nontrivial power. Therefore these two spaces is not homotopy equivalent. The basic tool for computing cohomology groups is the universal coefficient theorem, which turns the computation of cohomology groups into homology groups. So we can transparent computation methods for homology into cohomology contributing to the universal coefficient theorem. The computation for cohomology operations requires much more work.

The last classical one is called homotopy group π_n. It is the generalization of the fundamental group, $\pi_n(X)$, the n dimensional homotopy group. It is abelian when $n \geqslant 2$. Homotopy groups are very hard to compute due to the failure of excision. Even for the the simplest space S^i, we do not have a conclusion for its homotopy groups for arbitrary i. Tools for computing homotopy groups are various, such as Freudental suspension theorem, covering space, Hurewicz theorem, fiber bundles, fibrations and so on. Hurewicz theorem provides the relation between homotopy groups and homology groups. As for cohomology, there is a nice theorem saying that there are natural bijections $T : \langle X, K(G,n) \rangle \to \mathrm{H}^n(X; G)$ from the set of homotopy classes preserving the base-point from X to the Eilenberg-MacLane space $K(G,n)$ to the n-th cohomology group of X with coefficient group G. Fiber bundle $F \to E \to B$ provides the long exact sequence just like in homology for pairs $A \to X \to X \backslash A$. Freudental suspension theorem says the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is isomorphic when $i < 2n-1$ and X being (n-1)-connected CW complexes. This theorem tells us these maps will eventually be isomorphisms even there is no assumption on the connectivity of X.

$$\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots.$$

We call this eventually stable group the stable homotopy group of X, denoted by $\pi_i^s(X)$. The calculation of the i-th stable homotopy group $\pi_i^s = \pi_i^s(S^0)$ is an important problem. Serre used spectral sequence showing that π_i^s is always finite for $i > 0$, and Adams spectral sequence obtained the result of π_i^s for small i.

This article will be divided in two parts. The first part is the introduction to general spectral sequence, and use it to deduce some basic results as examples. The second part is about cohomology operations, including Steenrod operations and Adams operations. Adams spectral sequence will be the application of Steenrod operations in spectral sequence, and we will use it to calculate some famous examples. As I just said, these methods are tedious and hard to understand, but it worth the efforts. Once accepting them, one will get amazed by their magic power!

CHAPTER 2 MOTIVATIONS

As explained in chapter 1, algebraic topology translates a geometric problem into a homotopy theory problem, then using algebraic tools solving them. The first step is slightly easier then the second step. Starting with a general situation. Let $[X, Y]$ be the homotopy classes of maps between X and Y. One may ask whether a map $f : X \to Y$ is null homotopic or essential, that is, not null homotopic? One way to achieve this is applying reduced mod 2 cohomology $\tilde{H}^*(-, \mathbb{Z}_2)$. If f induces $f^* \in \mathrm{Hom}\,(\tilde{H}^*(Y; \mathbb{Z}_2), \tilde{H}^*(X; \mathbb{Z}_2))$ nontrivial, then f must be essential. But this method is too coarse. Consider the following example.

Let X, Y be S^1 and f be the squaring map. Cohomology with $\mathbb{Z}$ coefficients detects f is essential, with degree 2. But passing to $\mathbb{Z}_2$ coefficients, the reduced mod 2 cohomology becomes invalid. Fortunately, we have a chance to repair it.

Consider maps $X \xrightarrow{f} Y \to Y \cup_f CX$ and induces a long exact sequence in cohomology. If f induces 0 in mod 2 cohomology, then the exact sequence splits into a short exact sequence:

$$0 \leftarrow \tilde{H}^*(Y; \mathbb{Z}_2) \leftarrow \tilde{H}^*(Y \cup_f CX; \mathbb{Z}_2) \leftarrow \tilde{H}^*(SX; \mathbb{Z}_2) \leftarrow 0$$

Now if f is null homotopic, then $Y \cup_f CX \simeq Y \vee SX$. And this sequence splits. Hence $\tilde{H}^*(Y \cup_f CX; \mathbb{Z}_2) \cong \tilde{H}^*(Y \vee SX; \mathbb{Z}_2)$. Our chance is using the ring strucure of cohomology. Therefore, we have $\tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}_2) \cong \tilde{H}^*(S^1 \vee S^2; \mathbb{Z}_2)$ as ring isomorphism, contradiction.

Do not get satisfied too early. When passing to $Sf : S^2 \to S^2$, the modified method becomes invalid again, since suspension isomorphism $\tilde{H}^i(X; \mathbb{Z}_2) \cong \tilde{H}^{i+1}(SX; \mathbb{Z}_2)$ does not preserve the ring structure. But don't get frustrated, we can still repair our method, though it needs more efforts.

The ring structure is not enough to get over this obstruction. We need something called Steenrod algebra $\mathcal{A}_2$, which consists of Steenrod squares which are stable under suspension. The mod 2 cohomology, then becomes a $\mathcal{A}_2$-module. And this new structure can help us.

To summarize,

1. Consider $f^* \in \mathrm{Hom}\,(\tilde{H}^*(Y; \mathbb{Z}_2), \tilde{H}^*(X; \mathbb{Z}_2))$, and if it is zero,

2. Consider $\tilde{H}^*(Y \cup_f CX; \mathbb{Z}_2)$ as an element of $\mathrm{Ext}^1_{\mathcal{A}_2}(\tilde{H}^*(Y; \mathbb{Z}_2); \tilde{H}^*(SX; \mathbb{Z}_2))$. If it is unfortunately zero again, then we have some way to continuing such step, which leads to the Adams spectral sequences. Before introducing all concepts and main theories, let me explain two cases the method fails destined.

The first case is the suspension destroys f. Hence all we can detect using this method is the set $[S^k X, S^k Y]$ for large k. The Freudental suspension theorem implies that this group is stable for $k > \dim X + 2$. 2 ensures that this group is abelian. Denote this group as $\{X, Y\}$.

The second case is that this method can only detect mod2 phenomenon. For instance, $3f \simeq 0$, then it tells you f is null homotopic.

Fortunately, this method has these two blindness only. As it will explain in chapter 4. The next chapter introduces general spectral sequences, and the last two chapters give some interesting examples of its own, inspite of our original motivation.

CHAPTER 3 SPECTRAL SEQUENCES

Exact sequences are important tools in algebraic topology, but sometimes their complicated relations could not fit into exact sequences. Thus more tools will be needed. The spectral sequences arise in a natural way.

3.1 Bicomplexes

Bicomplexes provide a perfect stage for spectral sequences. So we will start with bicomplexes.

Definition 3.1: A *graded module* is an indexed collection $M = (M_p)_{p\in\mathbb{Z}}$ of R-modules (for some ring R), denoted by $M_\bullet$ for convenience.

Example 3.1:

1. A complex (C, ∂) is a graded module.
2. Homology $H_\bullet(C)$ of a complex is a graded module.

Definition 3.2: A *graded map of degree* a, denoted by $f : M \to N$, is a collection of maps $f = (f_p : M_p \to N_{p+a})_{p\in\mathbb{Z}}$ where M, N are graded modules and $a \in \mathbb{Z}$. Denote the degree of f by $\deg f = a$.

Example 3.2:

1. The differential of a complex is a graded map with degree -1.
2. A chain map $f : C \to C'$ is a graded map with $\deg f = 0$.

We can define $\mathrm{Hom}(M, N)$ to be

$$\mathrm{Hom}(M, N) = \bigcup_{a\in\mathbb{Z}}(\prod_p \mathrm{Hom}(M_p, N_{p+a}),$$

then graded modules over a fixed ring form a category.

M' is a submodule of M means $M_p{}' \subset M_p$ for all p. If M' is a submodule of M, then the quotient module $M/M' = (M_p/M_p{}')_{p\in\mathbb{Z}}$. It is obvious that both inclusions and natural quotient maps have degree 0. For graded map $f : M \to N$ with degree a, $\ker f = (\ker f_p)_{p\in\mathbb{Z}} \subset M$, $\mathrm{im} f = (\mathrm{im} f_{p-a})_{p\in\mathbb{Z}} \subset N$. Therefore, $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\mathrm{im} f = \ker g$.

Foe $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ a short exact sequence of complexes, the long exact sequence of their homology modules can be summarized as an *exact triangle*:

$$\begin{array}{ccc} H_\bullet(C) & \xrightarrow{\quad i_* \quad} & H_\bullet(C') \\ & {}_{\partial}\nwarrow \quad \swarrow_{j_*} & \\ & H_\bullet(C''), & \end{array}$$

where each vertex is a graded module, and maps are exactly graded maps: $\deg i_* = \deg j_* = 0$, and $\deg \partial = -1$.

Bigraded modules are something generalizing graded modules.

Definition 3.3: A *bigraded module* is a indexed collection

$$M = (M_{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$$

of $R-$modules, denoted by $M_{\bullet\bullet}$.

Definition 3.4: Suppose M and N are bigraded modules, and $(a, b) \in \mathbb{Z}\times\mathbb{Z}$. A *bigraded map of bidegree* (a, b), represent by $f : M \to N$, is a collection of maps $f = (f_{p,q} : M_{p,q} \to N_{p+a,q+b})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. Denote the *bidegree* of f by $\deg f = (a, b)$.

Just like graded modules, we can define morphisms between two bigraded modules M, N to form a category. The definition for submodules, quotient modules and exactness of bigraded modules is something analogue to graded modules. A *bicomplex* is a bigraded module with all maps being differentials.

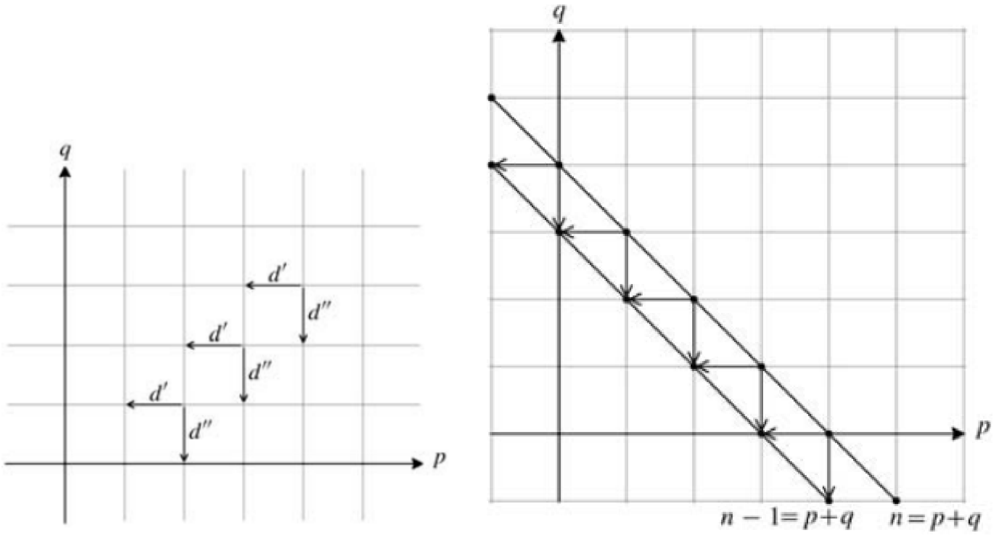
Definition 3.5: A *bicomplex* is an ordered triple (M, d', d''),where M is a bigraded module, $d', d'' : M \to M$ are differentials with $\deg d' = (-1, 0)$ and $\deg d'' = (0, -1)$, and

$$d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0.$$

A bicomplex M can be drawn in the $pq-$plane with $M_{p,q}$ lying on point (p, q). The rows $M_{*,q}$ and the columns $M_{p,*}$ are complexes. The equation $d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0$ says that each square *anticommutes*. (Fig. 3.1(a))

Remark 3.1: It doesn't matter one get confused about the anticommutativity at the first glance, since we can always transform a commutative bigraded module with differentials d', d'' into a bicomplex. All one needs to do is a *sign change*. Let $\Delta''_{p,q} = (-1)^p d''_{p,q}$. Kernels and images are not affected by changing signs, thus $\Delta''\Delta'' = 0$, the columns are still complexes. As for anticommutativity:

$$\begin{aligned} d'_{p,q-1}\Delta''_{p,q} + \Delta''_{p-1,q}d'_{p,q} &= (-1)^p d'_{p,q-1}d''_{p,q} + (-1)^{p-1}d''_{p-1,q}d'_{p,q} \\ &= (-1)^p(d'_{p,q-1}d''_{p,q} - d''_{p-1,q}d'_{p,q}) \\ &= 0. \end{aligned}$$



a) Bicomplex. b) Total complex.

Figure 3-1 [1]

Therefore, (M, d', Δ'') is a bicomplex.

Definition 3.6: If M is a bicomplex, its *total complex*, Tot(M), is the complex whose nth term

$$Tot(M)_n = \bigoplus_{p+q=n} M_{p,q}$$

and with differentials D_n : Tot$(M)_n \to$ Tot$(M)_{n-1}$ given by

$$D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q})$$

(Fig. 3.1(b)).

The total complex (Tot(M),D) is indeed a complex. Since im$d'_{p,q} \subset M_{p-1,q}$ and im$d'_{p,q}{}' \subset M_{p,q-1}$; no matter in which case, the sum of indices will be $p + q - 1 = n - 1$, thus im$D \subset$ Tot$(M)_{n-1}$.

As for D is a differential:

$$\begin{aligned} DD &= \sum_{p,q}(d' + d'')(d' + d'') \\ &= \sum d'd' + \sum(d'd'' + d''d') + \sum d''d'' \\ &= 0. \end{aligned}$$

Spectral sequences can be used to compute the homology of a total complex Tot(M). Before the really computation, we shall see some example first.

Example 3.3:

1. Let R be a ring, and suppose

$$\mathbf{A} =\to A_p \xrightarrow{\Delta'_p} A_{p-1} \to \cdots \to A_0 \to 0$$

and

$$\mathbf{B} =\to B_q \xrightarrow{\Delta''_q} B_{q-1} \to \cdots \to B_0 \to 0$$

are complexes. Define (M, d', d'') by

$$M_{p,q} = A_p \otimes_R B_q,\ d'_{p,q} = \Delta'_p \otimes 1_{B_q}, \text{ and } d''_{p,q} = (-1)^p 1_{A_p} \otimes \Delta''_q.$$

This is a bicomplex, the total complex denoted by $\mathrm{Tot}(M) = \mathbf{A} \otimes \mathbf{B}$ is called the *tensor product of complexes*.

$$(\mathbf{A} \otimes \mathbf{B})_n = \bigoplus_{p+q=n} A_p \otimes_R B_q,$$

and $D_n : (\mathbf{A} \otimes \mathbf{B})_n \to (\mathbf{A} \otimes \mathbf{B})_{n-1}$ is given by

$$D_n : a_p \otimes b_q \mapsto \Delta' a_p \otimes b_q + (-1)^p a_p \otimes \Delta''_q b_q.$$

Definition 3.7: A *first quadrant bicomplex* is a bicomplex with $M_{p,q} = 0$ whenever p or q is negative.

2. Let A, B be $R-$modules, and let $\mathbf{P}_A$, $\mathbf{Q}_B$ be deleted projective resolutions. By 1, we yield a bicomplex, and this example will help us to prove the Tor functor is independent of which variable performing resolution.
3. The *Eilenberg-Zilber Theorem* says that

$$H_n(X \times Y) \cong H_n(\mathbf{S}_\bullet(X) \otimes_{\mathbb{Z}} \mathbf{S}_\bullet(Y)),$$

where X and Y are topological spaces and $\mathbf{S}_\bullet(X)$ is the singular complex of X. And we can use spectral sequence to prove the Kunneth formula.

3.2 Exact Couples

Suppose we have a filtration $(F^pC)_{p\in\mathbb{Z}}$ of complex C, that is the commutative diagram with the vertical maps being inclusions and the horizontal maps differentials.

$$\begin{array}{rccccccc}
C : & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow \\
 & & \uparrow & & \uparrow & & \uparrow & \\
F^pC : & \longrightarrow & F^p_{n+1} & \longrightarrow & F^p_n & \longrightarrow & F^p_{n-1} & \longrightarrow \\
 & & \uparrow & & \uparrow & & \uparrow & \\
F^{p-1}C : & \longrightarrow & F^{p-1}_{n+1} & \longrightarrow & F^{p-1}_n & \longrightarrow & F^{p-1}_{n-1} & \longrightarrow
\end{array}$$

Abbreviate F^pC to F^p. For each fiexed p, there is a short exact sequence of complexes,

$$0 \longrightarrow F^{p-1} \xrightarrow{i^{p-1}} F^p \xrightarrow{j^p} F^p/F^{p-1} \longrightarrow 0$$

that gives a long exact sequence

$$\longrightarrow H_n(F^{p-1}) \xrightarrow{\alpha} H_n(F^p) \xrightarrow{\beta} H_n(F^p/F^{p-1}) \xrightarrow{\gamma}$$

$$H_{n-1}(F^{p-1}) \xrightarrow{\alpha} H_{n-1}(F^p) \xrightarrow{\beta} H_{n-1}(F^p/F^{p-1}) \longrightarrow,$$

where $\alpha = i_*^{p-1}$, $\beta = j_*^p$, and $\gamma = \partial$. Let $p+q=n$, we can rewrite this sequence as

$$\longrightarrow H_{p+q}(F^{p-1}) \xrightarrow{\alpha} H_{p+q}(F^p) \xrightarrow{\beta} H_{p+q}(F^p/F^{p-1}) \xrightarrow{\gamma}$$

$$H_{p+q-1}(F^{p-1}) \xrightarrow{\alpha} H_{p+q-1}(F^p) \xrightarrow{\beta} H_{p+q-1}(F^p/F^{p-1}) \longrightarrow.$$

Observing that there are two types of homology groups: $H_n(F^p)$ and $H_n(F^p/F^{p-1})$. Define

$$D = (D_{p,q}), \text{ where } D_{p,q} = H_{p+q}(F^p),$$
$$E = (E_{p,q}), \text{ where } E_{p,q} = H_{p+q}(F^p/F^{p-1}).$$

With these notation, we can summarize all long exact sequences here, i.e. for each p, as an *exact couple*[2].

Definition 3.8: An *exact couple* is an ordered tuple $(D, E, \alpha, \beta, \gamma)$, with D, E beging bigraded modules, α, β, γ being bigraded maps. And at each vertex, maps are exact: $\ker\alpha = \mathrm{im}\gamma$, $\ker\beta = \mathrm{im}\alpha$, $\ker\gamma = \mathrm{im}\beta$.

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta} \\ & E & \end{array}$$

Using the notation above, every filtration $(F^pC)_{p\in\mathbb{Z}}$ of a complex C will provide an exact couple

$$\begin{array}{ccc} D & \xrightarrow{\alpha(1,-1)} & D \\ {\scriptstyle\gamma(-1,0)}\nwarrow & & \swarrow{\scriptstyle\beta(0,0)} \\ & E & \end{array}$$

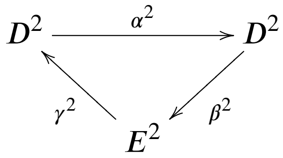
Definition 3.9: A *differential bigraded module* is an ordered pair (M, d), where M is a bigraded module and $d : M \to M$ is a differential. Suppose $\deg d = (a, b)$, the *homology*

$H(M, d)$ is then a bigraded module with p, q term

$$H(M, d)_{p,q} = \frac{\ker d_{p,q}}{\mathrm{im} d_{p-a,q-b}}.$$

A bicomplex (M, d', d'') yields two differential bigraded modules, (M, d') and (M, d'').

Proposition 3.1: If $(D, E, \alpha, \beta, \gamma)$ is an exact couple, then $d^1 = \beta\gamma$ is a differential $d^1 : E \to E$, and there is an exact couple $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$ called the *derived couple*, with $E^2 = H(E, d^1)$.



Proof: We just look into the case of exact couple coming from the filtration of a complex, where $\deg \alpha = (1, -1)$, $\deg \beta = (0, 0)$, $\deg \gamma = (-1, 0)$. First, verify that d^1 is a differential: $d^1 d^1 = \beta(\gamma\beta)\gamma = 0$, by the exactness of the original couple. $\deg d^1 = (-1, 0)$.

Define $E^2 = H(M, d^1)$. Thus, $E^2_{p,q} = \ker d^1_{p,q} / \mathrm{im} d^1_{p+1,q}$.

Define $D^2 = \mathrm{im}\ \alpha \subset D$. Thus, $D^2_{p,q} = \mathrm{im}\ \alpha_{p-1,q+1} \subset D_{p,q}$.

We now define maps.

Let $\alpha^2 : D^2 \to D^2$ to be the restriction $\alpha | D^2$. Clearly, $\deg \alpha^2 = \deg \alpha = (1, -1)$. If $x \in D^2_{p,q}$, then $x = \alpha u$ for $u \in D_{p-1,q+1}$, and

$$\alpha^2_{p,q} : x = \alpha u \mapsto \alpha x = \alpha\alpha u$$

Define $\beta^2 : D^2 \to E^2$ as follows. If $y \in D^2_{p,q}$, then $y = \alpha v$ for some $v \in D_{p-1,q+1}$ and βv is a cycle for $d^1 \beta v = \beta(\gamma\beta)v = 0$. Hence

$$\beta^2 : y \mapsto \mathrm{cls}(\beta v).$$

we shall verify β^2 is well-defined. Suppose $y = \alpha v'$, then $v - v' \in \ker \alpha = \mathrm{im}\ \gamma$. Thus there is $\omega \in D_{p+1,q-1}$ with $\gamma\omega = v - v'$. Therefore $\beta(v - v') = \beta\gamma\omega = d^1\omega$ is a boundary. Note that $\deg \beta^2 = (-1, 1)$

Define $\gamma^2 : E^2 \to D^2$ as follows. Let $\mathrm{cls}(z) \in E^2_{p,q}$, thus $d^1(z) = \beta\gamma z = 0$. $\gamma z \in \ker \beta = \mathrm{im}\ \alpha$. Define γ^2 by

$$\gamma^2 : \mathrm{cls}\ (z) \mapsto \gamma z.$$

If ω is a boundary, that is $\omega \in \mathrm{im}\ d^1_{p+1,q-1}$, $\omega = d^1 x = \beta\gamma x$. Then $\gamma^2 \mathrm{cls}(\omega) = \gamma\beta\gamma x = 0$. Hence γ^2 is independent of the choice of representatives. Note that $\deg \gamma^2 = (-1, 0)$.

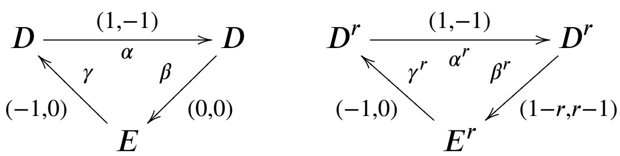
What remains is to verify the exactness at each vertex. First of all, adjacent composites are 0.

$$\beta^2\alpha^2 \,:\, x = \alpha u \mapsto \alpha\alpha u \mapsto \mathrm{cls}\,(\beta\alpha u) = 0.$$
$$\gamma^2\beta^2 \,:\, x = \alpha u \mapsto \mathrm{cls}\,(\beta u) \mapsto \gamma\beta u = 0.$$
$$\alpha^2\gamma^2 \,:\, \mathrm{cls}\,(z) \mapsto \gamma z \mapsto \alpha\gamma z = 0.$$

$\ker\alpha^2 \subset \mathrm{im}\,\gamma^2$: If $x \in \ker\alpha^2$, then $\alpha x = 0 \in D^2$. Hence $x \in \ker\alpha = \mathrm{im}\,\gamma$, thus there is a $y \in E$ with $x = \gamma y$. And $x \in D^2 = \mathrm{im}\,\alpha = \ker\beta$, $\beta x = \beta\gamma y = d^1 y = 0$, hence $\mathrm{cls}(y) \in E^2$ with $x = \gamma y = \gamma^2\mathrm{cls}\,(y) \in \mathrm{im}\,\gamma^2$.
$\ker\beta^2 \subset \mathrm{im}\,\alpha^2$: If $x \in \ker\beta^2$, $x = \alpha v$ with $\beta v \in \mathrm{im}\,d^1$. That is $\beta v = d^1\omega = \beta\gamma\omega$. Hence $v - \gamma\omega \in \ker\beta = \mathrm{im}\,\alpha = D^2$. Notice that $\alpha^2(v - \gamma\omega) = \alpha v - \alpha\gamma\omega = \alpha v = x$, therefore $x \in \mathrm{im}\,\alpha^2$.
$\ker\gamma^2 \subset \mathrm{im}\,\beta^2$: If $\mathrm{cls}(z) \in \ker\gamma^2$, $\gamma z = 0$, thus $z \in \ker\gamma = \mathrm{im}\,\beta$, $z = \beta v$ for some $v \in D$. Observe that $\alpha v \in D^2$ so, $\mathrm{cls}(z) = \beta^2(\alpha v) \in \mathrm{im}\,\beta^2$. ∎

Definition 3.10: The *rth derived couple* of an exact couple $(D, E, \alpha, \beta, \gamma)$ inductively: its $(r+1)$st derived couple $(D^{r+1}, E^{r+1}, \alpha^{r+1}, \beta^{r+1}, \gamma^{r+1})$ is the derived couple of $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$, the rth derived couple.

Theorem 3.1: Let $(D, E, \alpha, \beta, \gamma)$ be the couple of a filtration (F^pC).



Then:

1. the bigraded maps α^r, β^r, γ^r have bidegrees $(1,-1), (1-r, r-1), (-1, 0)$, respectively;
2. the differential $d^r = \beta^r\gamma^r$ has bidegree $(-r, r-1)$;
3. $E^{r+1} = H(E^r, d^r)$;
4. $D^r_{p,q} = \mathrm{im}\,(\alpha_{p-1,q+1})(\alpha_{p-2,q+2}) \cdots (\alpha_{p-r+1,q+r-1})$; in particular, for the exact couple in the beginning of this section,

$$D^r_{p,q} = \mathrm{im}\,(i^{p-1}i^{p-2}\cdots i^{p-r+1})_* \,:\, H_n(F^{p-r+1}) \to H_n(F^p).$$

Proof: The proof of this theorem is trivial by induction. Also notice that $d^1_{p,q} : E_{p,q} \to E_{p-1,q}$ is the last circumstance is the connecting homomorphism ∂

$$H_{p+q}(F^p/F^{p-1}) \to H_{p+q-1}(F^{p-1}/F^{p-2})$$

arising from $0 \to F^{p-1}/F^{p-2} \to F^p/F^{p-2} \to F^p/F^{p-2} \to 0$. ∎

Definition 3.11: A *spectral sequence* is a collection $(E^r, d^r)_{r\geqslant 1}$ of differential bigraded modules such that $E^{r+1} = H(E^r, d^r)$ for all r. By previous discussion, every filtration of a complex provides a spectral sequence.

3.3 Convergence

Theorem 3.1 states that a filtration of complex C provides a spectral sequence, but what is the connection between the E^r term of the spectral sequence and the homology of $H_\bullet(C)$?

If (E^r, d^r) is a spectral sequence, then $E^2 = H_\bullet(E^1, d^1)$ is a subquotient of E^1: hence, $E^2 = Z^2/B^2$, where

$$B^2 \subset Z^2 \subset E^1.$$

And Z^3, B^3 can be viewed as quotients $B^3/B^2 \subset Z^3/B^2 \subset Z^2/B^2 = E^2$, so that

$$B^2 \subset B^3 \subset Z^3 \subset Z^2 \subset E^1.$$

Continuing such steps, for each r, there is a chain

$$B^2 \subset \cdots \subset B^r \subset Z^r \subset \cdots \subset Z^2 \subset E^1.$$

Definition 3.12: Given a spectral sequence E^r, d^r, define $Z^\infty = \cap_r Z^r$ and $B^\infty = \cup_r B^r$. Then $B^\infty \subset Z^\infty$, the *limit term* is defined by

$$E^\infty_{p,q} = Z^\infty_{p,q} \,/\, B^\infty_{p,q}.$$

Clearly, $E^{r+1} = E^r$ iff $Z^{r+1} = Z^r$ and $B^{r+1} = B^r$; and if $E^{r+1} = E^r$ for all $r \geqslant s$, then $E^s = E^\infty$.

Definition 3.13: Let $(F^p(C)$ be a filtration of a complex C. Let $i^p : F^p \to C$ be the inclusions, define the induced filtration of $H_n(C)$ to be

$$\Phi^p H_n(C) = \operatorname{im} i^p_*.$$

Definition 3.14: A filtration $(F^p M)$ of a graded module $M = (M_n)$ is *bounded* if , for any n, we can find integers $s = s(n)$, $t = t(n)$ such that

$$F^s M_n = 0 \text{ and } F^t M_n = M_n.$$

If F^p is a bounded filtration of a complex, the induced filtration on homology of that complex will be bounded as well. Moreover, their bounds are equal.

Definition 3.15: A spectral sequence $(E^r, d^r)_{r\geqslant 1}$ *converges* to H, a graded module, denoted by

$$E^2_{p,q} \underset{p}{\Rightarrow} H_n,$$

if there is some *bounded* filtration $(\Phi^p H_n)$ of H with

$$E^\infty_{p,q} \cong \Phi^p H_n \,/\, \Phi^{p-1} H_n$$

for any $p + q = n$.

Theorem 3.2: [3] Let $(F^p C)$ be a bounded filtration, and $(E^r, d^r)_{r \geqslant 1}$ be the associated spectral sequence. Then

1. for any p, q, $E^\infty_{p,q} = E^r_{p,q}$ for large r, depending on p, q,
2. $E^2_{p,q} \underset{p}{\Rightarrow} H_n(C)$.

Proof:

1. If p is large, that is $p > t(n)$, then $F^{p-1} = F^p$, and $F^p/F^{p-1} = 0$. So that $E_{p,q} = H_{p+q}(F^p/F^{p-1}) = 0$. Since $E^r_{p,q}$ is a subquotient of $E_{p,q}$, we have $E^r_{p,q} = 0$ for every r. If $p < s(n)$, we have $F^p = 0$, therefore $E^r_{p,q} = 0$ for every r.

   Then considering the differential d^r, which has degree $(-r, r-1)$. For any fiexed (p, q), we can choose a sufficient large r, such that $p-r < s(n)$ and $p+r > t(n)$. In this circumstance, $d^r_{p,q} = d^r_{p+r,q-r+1} = 0$, thus $E^{r+1}_{p,q} = E^r_{p,q}$, which yields $E^r_{p,q} = E^\infty_{p,q}$.
2. Writing • for all second indices and observe the first subscript.Look into the exact sequence coming from the rth couple:

   $$D^r_{p+r-2,\bullet} \xrightarrow{\alpha^r} D^r_{p+r-1,\bullet} \xrightarrow{\beta^r} E^r_{p,q} \xrightarrow{\gamma^r} D^r_{p-1,q}. \qquad (3\text{-}1)$$

   The module

   $$D^r_{p,q} = \operatorname{im}\,(i^{p-1} i^{p-2} \cdots i^{p-r+1})_* : H_n(F^{p-r+1}) \to H_n(F^p).$$

   Replacing p first by $p+r-1$ then by $p+r-2$, we have

   $$D^r_{p+r-1,\bullet} = \operatorname{im}\,(i^{p+r-2} \cdots i^p)_* \subset H_n(F^{p+r-1})$$
   $$D^r_{p+r-2,\bullet} = \operatorname{im}\,(i^{p+r-3} \cdots i^{p-1})_* \subset H_n(F^{p+r-2}).$$

   For large r, $F^{p+r-1} = F^t = C$, and the composition of the inclusions is just the inclusion $i^p : F^p \to C$. Therefore, $D^r_{p+r-1,\bullet} = \operatorname{im} i^p_* = \Phi^p H_n$ and $D^r_{p+r-2,\bullet} = \operatorname{im} i^{p-1}_* = \Phi^{p-1} H_n$. Hence we can rewrite (3-1) in the following:

   $$\Phi^{p-1} H_n \to \Phi^p H_n \to E^r_{p,q} \to D^r_{p-1,q},$$

where the first map is inclusion. If $D^r_{p-1,q} = 0$, then for sufficient large r,

$$\Phi^p H_n / \Phi^{p-1} H_n \cong E^r_{p,q} = E^\infty_{p,q},$$

and we are done. But $D^r_{p-1,q} = \mathrm{im}\,(H_n(F^{p-r}) \to H_n(F^{p-1}))$, which is zero for $H_n(F^{p-r}) = 0$ when r sufficient large.

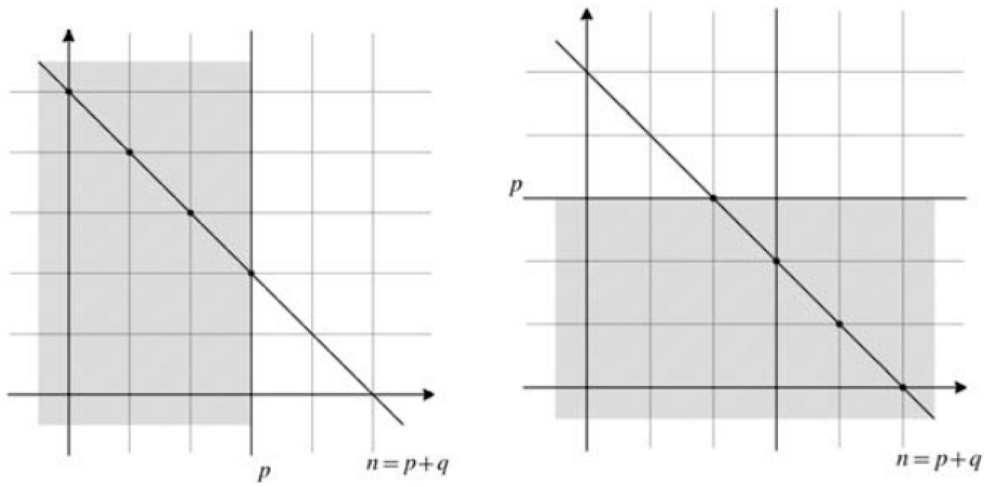
∎

3.4 Homology of the Total Complex

Now let us turn to the calculation of total complexes arising from bicomplexes. For a bicomplex (M, d', d''), $\mathrm{Tot}(M)$ can be filtered in two different ways. The *first filtration* of $\mathrm{Tot}(M)$ is $({}^{\mathrm{I}}F^p)$, where

$$\begin{aligned}({}^{\mathrm{i}}F^p)_n &= \bigoplus_{i \leqslant p} M_{i,n-i} \\ &= \cdots \oplus M_{p-2,q+2} \oplus M_{p-1,q+1} \oplus M_{p,q}.\end{aligned}$$

The nth term of it is clearly the direct sum of all $M_{i,n-i}$ on the left of a vertical line.



a) First filtration b) Second filtration

Figure 3-2 [1]

The *second filtration* of $\mathrm{Tot}(M)$ is $({}^{\mathrm{II}}F^p)$, where

$$\begin{aligned}({}^{\mathrm{II}}F^p)_n &= \bigoplus_{j \leqslant p} M_{n-j,j} \\ &= \cdots \oplus M_{q+2,p-2} \oplus M_{q+1,p-1} \oplus M_{q,p},\end{aligned}$$

The nth term of it is clearly the direct sum of all $M_{i,n-i}$ below a horizontal line.

From now on, suppose the bicomplex is first quadrant. Let $({}^{\mathrm{I}}E^r)$ and $({}^{\mathrm{II}}E^r)$ be the spectral sequences associated with the two filtrations of $\mathrm{Tot}(M)$. By previous theorem 3.2, we have the following:

Corollary 3.1: The two filtrations are bounded and

1. For any p, q, $({}^{\mathrm{I}}E^{\infty}_{p,q}) = ({}^{\mathrm{I}}E^r_{p,q})$ and $({}^{\mathrm{II}}E^{\infty}_{p,q}) = ({}^{\mathrm{II}}E^r_{p,q})$ for large r depending on p, q.
2. $({}^{\mathrm{I}}E^2_{p,q}) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M))$ and $({}^{\mathrm{II}}E^2_{p,q}) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M))$.

What makes spectral sequences so useful is that the E^2 page of a spectral sequence arising from a bicomplex is computable. We will compute $({}^{\mathrm{I}}E^2_{p,q})$ below, hence throwing away the prescript I in the following argument.

$E_{p,q} = H_n(F^p/F^{p-1})$, notice that the nth term of F^p/F^{p-1} is just $M_{p,q}$ by definition. The differential $(F^p/F^{p-1})_n \to (F^p/F^{p-1})_{n-1}$ is

$$\overline{D_n} : a_n + (F^{p-1})_n \mapsto D_n a_n + (F^{p-1})_{n-1},$$

where $a_n \in (F^p)_n$; we may assume $a_n \in M_{p,q}$. Now $D_n a_n = (d'_{p,q} + d''_{p,q})a_n \in M_{p-1,q} \oplus M_{p,q-1}$. But $M_{p-1,q} \subset (F^{p-1})_n$, so that $D_n a_n \equiv d''_{p,q} a_n \mod (F^{p-1})_{n-1}$. Thus only d'' survives in F^p/F^{p-1}. To be precise,

$$H_n(F^p/F^{p-1}) = \frac{\ker \overline{D_n}}{\mathrm{im}\, \overline{D_{n+1}}} \cong \frac{\ker d''_{p,q}}{\mathrm{im}\, d''_{p,q+1}} = H_q(M_{p,*}),$$

where ($M_{p,*}$ is the pth column of M which is a complex with differentials d''. And there are horizontal maps d' survive. Hence consider the qth row,

$$\dots, H_q(M_{p-1,*}), H_q(M_{p,*}), H_q(M_{p+1,*}), \dots,$$

which is complex whose differentials are induced by d'. So far, we already define another bigraded module with p, q term denoted by $H'_p H''_q(M)$ being first taking homology qth in pth column, then taking pth homology in qth row. Briefly speaking, first taking homology vertically, then horizontally.

Definition 3.16: For bicomplex (M, d', d''), the bigraded module whose (p, q) term being $H'_p H''_q(M)$ is called the *first iterated homology*.

Next theorem leads to a miracle.

Theorem 3.3: If M is a first quadrant bicomplex, then

$$({}^{\mathrm{I}}E^1_{p,q}) = H_q(M_{p,*})$$

$$({}^{\mathrm{I}}E^2_{p,q}) = H'_p H''_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Hence we can compute the E^2 page through the first iterated homology.

Proof: Only thing to verify is the second statement. Omit the prescript I for convenience. We show that $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ takes cls$z \mapsto \mathrm{cls}(\overline{d'}z) \in H'_p H''_q(M)$ is the same as differentials defined in the first iterated homology. Then the consequence follows. As $d^1 : H_{p+q}(F^p/F^{p-1}) \to H_{p+q-1}(F^{p-1}/F^{p-2})$ is the connecting homomorphism, we have a diagram in chain complexes:

$$\begin{array}{ccccccc} & & M_{p-1,q+1} \oplus M_{p,q} & \xrightarrow{j} & M_{p,q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{D}} & & & & \\ 0 & \longrightarrow & M_{p-1,q} \xrightarrow{i} M_{p-1,q} \oplus M_{p,q-1}. & & & & \end{array}$$

where $\overline{D} : (a_{p-1,q}, a_{p,q}) \mapsto (d''a_{p-1,q} + d'a_{p,q} \ , \ d''a_{p,q})$. Let $z \in M_{p,q}$ be a cycle; that is, $d''_{p,q}z = 0$. Choose $j^{-1}z = (0, z)$, so that $\overline{D}(0, z) = (d'_{p,q}z, 0)$ for $d'' = 0$.

$$d^1\mathrm{cls}(z) = \mathrm{cls}(i^{-1}\overline{D}j^{-1}z) = \mathrm{cls}(\overline{d'}z) \in H'_p H''_q(M).$$

∎

There is a dual version for the second filtration.

Definition 3.17: For a bicomplex (M, d', d''), the bigraded module whose (p, q) term is $H''_p H'_q(M)$ is called its *second iterated homology*. That is taking homology horizontally at pth row first, then vertically at qth column. Notice that the indices in this circumstance are interchanged.

Theorem 3.4: If M is a first quadrant bicomplex, then

$$({}^{\mathrm{II}}E^1_{p,q}) = H_q(M_{*,p})$$

$$({}^{\mathrm{II}}E^2_{p,q}) = H''_p H'_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Even both two spectral sequences converge to $H_n(\mathrm{Tot}(M))$, there may not be isomorphisms ${}^{\mathrm{I}}E^\infty_{p,q} \cong {}^{\mathrm{II}}E^\infty_{p,q}$ and the induced filtrations on H_n from $({}^{\mathrm{i}}F^p)$ and from $({}^{\mathrm{ii}}F^p)$ need not be the same. The following situation usually happens, that is, there are lots of 0 on the E^2 page.

Definition 3.18: A spectral sequence *collapses* on the p-axis if $E^2_{p,q} = 0$ for all $q \neq 0$; *collapses* on q-axis if $E^2_{p,q} = 0$ for each $p \neq 0$.

By arguing on the factor modules, one finds if a first quadrant spectral sequence converges, then

1. If it collapse on either axis, then $E^\infty_{p,q} = E^2_{p,q}$ for all p, q.
2. If it collapse on the p-axis, then $H_n(\mathrm{Tot}(M)) \cong E^2_{n,0}$.
3. If it collapse on the q-axis, then $H_n(\mathrm{Tot}(M)) \cong E^2_{0,n}$.

In homological algebra, there are two important functors Tor_n^R and Ext_n^R, the left derived functor of tensor and the right derived functor of Hom. These functors are actually bifunctors and independent of the variable resolved. We can use spectral sequence to show this property.

Example 3.4: Recall the notation:

$$\mathrm{Tor}_n^R(A,B) = H_n(\mathbf{P}_A \otimes_R B) \text{ and } \mathrm{tor}_n^R(A,B) = H_n(A \otimes_R \mathbf{Q}_B),$$

where $\mathbf{P}_A$ and $\mathbf{Q}_B$ are deleted projective resolutions of A and B respectively. Then

$$\mathrm{Tor}_n^R(A,B) = H_n(\mathbf{P}_A \otimes_R B) \cong H_n(\mathbf{P}_A \otimes \mathbf{Q}_B) \cong H_n(A \otimes_R \mathbf{Q}_B) = \mathrm{tor}_n^R(A,B)$$

Suppose we have bicomlex (M, d', d'') in section 3.1 whose total complex is $\mathbf{P}_A \otimes \mathbf{Q}_B$ and compute it with the first iterated homology. E^1 is $H_q''(M_{p,*})$, the qth homology of the pth column

$$M_{p,*} = \to P_p \otimes Q_{q+1} \to P_p \otimes Q_q \to\to P_p \otimes Q_{q-1} \to .$$

This sequence is exact for $q > 0$ because that P_p is projective, thus $H_q(M_{p,*}) = 0$ for $q > 0$. When $q = 0$, $Q_1 \to Q_0 \to B \to 0$ is exact respect to the functor $P_p \otimes \bullet$. Therefore, $H_q(M_{p,*}) = P_p \otimes B$. To sum up,

$${}^{\mathrm{I}}E^1_{p,q} = \begin{cases} 0, & \text{if } q > 0, \\ P_p \otimes B, & \text{if } q = 0. \end{cases}$$

Therefore, the spectral sequence collopses on the p-axis.

$${}^{\mathrm{I}}E^2_{p,q} = H_p' H_q''(M) = \begin{cases} 0, & \text{if } q > 0, \\ H_p(\mathbf{P}_A \otimes B), & \text{if } q = 0. \end{cases}$$

Thus the previous observation yields

$$H_n(\mathbf{P}_A \otimes \mathbf{Q}_B) = H_n(\mathrm{Tot}(M)) \cong {}^{\mathrm{I}}E^2_{n,0} \cong H_n(\mathbf{P}_A \otimes B).$$

A similar argument using the second iterated homology gives

$${}^{\mathrm{II}}E^2_{p,q} = H_p'' H_q'(M) = \begin{cases} 0 & \text{if } p > 0 \\ H_q(A \otimes \mathbf{Q}_B) & \text{if } p = 0. \end{cases}$$

Thus this spectral sequence collapse on p-axis,

$$H_n(\mathbf{P}_A \otimes \mathbf{Q}_B) = H_n(\mathrm{Tot}(M)) \cong {}^{\mathrm{II}}E^2_{0,n} \cong H_n(A \otimes \mathbf{Q}_B),$$

which proves our statement.

The Ext version will focus on a third quadrant bicomplex, and the argument is the same as it in the first quadrant.

3.5 The Serre Spectral Sequence

Suppose $\pi : X \to B$ is a filtration. The base space B is path connected, and has a cell structure. We can construct a filtration of X by the subspaces $X_p = \pi^{-1}(B^p)$, B^p being the $p-$skeleton of B. Since (B, B^p) is $p-$connected, (X, X_p) is $p-$connected as well for the homotopy lifting property, the inclusion $X_p \hookrightarrow X$ induces an isomorphism on H_n with any coefficient G. Together with $X_p =$ when $p < 0$, we obtain a bounded filtration of chain complex on X.

The E^1 term consists of $E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$, which is nonzero only when $p, q \geqslant 0$. Hence the spectral sequence is a first quadrant one. Together with the argument above, this spectral sequence converges to $H_*(X; G)$.

Theorem 3.5 (Serre Spectral Sequence[4]): Suppose we have a fibration $F \to X \to B$ with B path-connected. Then the corresponding spectral sequence converges, with

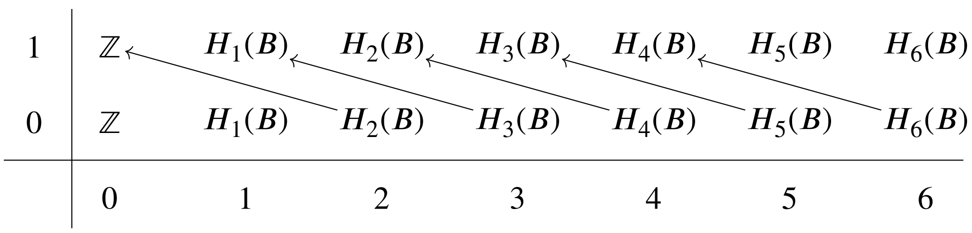
$$H_p(B; H_q(F; G)) \cong E^2_{p,q} \underset{p}{\Rightarrow} H_n(X; G),$$

if $\pi_1(B)$ acts trivially on $H_*(F; G)$.

The proof of this theorem is a quite long story, readers can find it in[5].

Example 3.5: From the preceding theorem, we can compute the homology of $K(\mathbb{Z}, 2)$. Firstly, we consider $\Omega B \to P \to B$ as a pathspace filtration of B being a $K(\mathbb{Z}, 2)$. $\Omega K(\mathbb{Z}, \mathbb{2})$ is a $K(\mathbb{Z}, \mathbb{1})$ wich can be identified with S^1. P is the pathspace of B which is contractible. Then the spectral sequence is in the following form.

E_2 page for $H_*(K(\mathbb{Z}, 2))$



$E_2^{p,q} = H_p(K(\mathbb{Z}, 2); H_q(S^1))$, and the differentials d^2 are indicated in the figure. Since P is contractible, $H^n(P) = 0$ for all $n \neq 0$, and d^i are all zero maps for $i \geqslant 3$, we know that $E^3_{p,q} = E^\infty_{p,q} = 0$ for $p, q > 0$. Therefore, all the differentials d^2 must be isomorphisms, which yields $H_{2n} = \mathbb{Z}$ and $H_{2n+1} = 0$.

There is an analogous Serre spectral sequence in cohomology, which is more powerful contributing to the extra structure on it.

Theorem 3.6: [6] Let $F \to X \to B$ be a filtration. Let B be a path-connected CW complex, then there is a spectral sequence converges to $H^n(X;G)$ with

$$E_2^{p,q} \cong H^p(B; H^q(F;G)).$$

if $\pi_1(B)$ acting trivially on cohomology $H^*(F;G)$.

The cohomological version of Serre spectral sequences have cup products structure. That is a bilinear map:

$$E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t} \tag{3-2}$$

satisfying:

1. d_r satisfies $d(xy) = d(x)y + (-1)^{p+q}xd(y)$ for $x \in E_r^{p,q}$. This implies the product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ induces a product $E_{r+1}^{p,q} \times E_{r+1}^{s,t} \to E_{r+1}^{p+s,q+t}$, and this is exactlythe product for E_{r+1}.
2. The product in E_2 page is $(-1)^{qs}$ times the standard cup product

$$H^p(B; H^q(F;R)) \times H^s(B; H^t(F;R)) \to H^{p+s}(B; H^{q+t}(F;R))$$

   sending (ϕ, ψ) to $\phi \smile \psi$, the coefficients are combined through the cup product in $H^*(F;R)$.
3. The cup product in $H^*(X;R)$ restricts to maps $F_p^m \times F_s^n \to F_{p+s}^{m+n}$. These induce quotient maps $F_p^m/F_{p+1}^m \times F_s^n/F_{s+1}^n \to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$ that coincide with the products $E_\infty^{p.m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

The tedious proof of statements above will not be covered. But with these properties in mind, we can start to compute something interesting.

Example 3.6: We can calculate the cohomology of $K(\mathbb{Z},2)$ through the product structure. Again, starting with the pathspace filtration $K(\mathbb{Z},1) \to P \to K(\mathbb{Z},2)$.

E_2 page for $H^*(K(\mathbb{Z},2))$

1	$\mathbb{Z}a$	0	$\mathbb{Z}ax_2$	0	$\mathbb{Z}ax_4$	0	$\mathbb{Z}ax_6$
0	$\mathbb{Z}1$	0	$\mathbb{Z}x_2$	0	$\mathbb{Z}x_4$	0	$\mathbb{Z}x_6$
	0	1	2	3	4	5	6

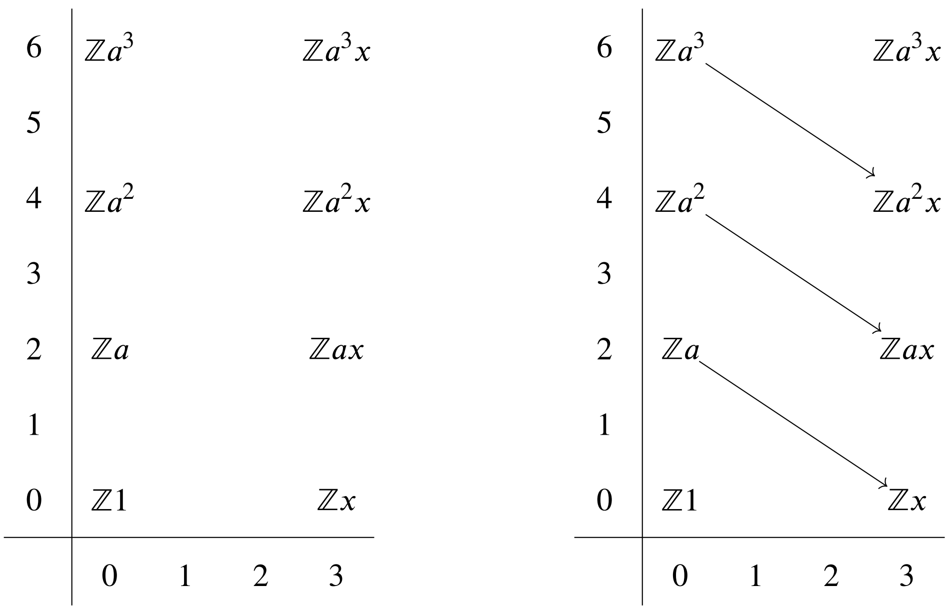
$E_2^{p,q} = H^p(K(\mathbb{Z},2); H^q(S^1))$. a and x_i are generators of $E_2^{0,i} = \mathbb{Z}$ and $E_2^{i,0} = \mathbb{Z}$. The differentials in this chart must be isomorphisms since P is contractible, and all terms except $\mathbb{Z}1$ disappear in E_∞. Hence we may regard $x_2 = d_2 a$. $x_{2i+2} = d_2(ax_{2i}) = (d_2 a)x_{2i} \pm a(d_2 x_{2i}) = (d_2 a)x_{2i} = x_2 x_{2i}$. This implies that $H^*(K(\mathbb{Z},2);\mathbb{Z}) = \mathbb{Z}[x_2]$ is a polynomial ring.

3.6 Computing $\pi_4(S^3)$

In this short section, we can calculate the p-torsion of $\pi_i(S^3)$. It is 0 when $i < 2p$ and $\mathbb{Z}_p$ when $i = 2p$.

As in previous section, we can find a map $S^3 \to K(\mathbb{Z}, 3)$ which is isomorphic on π_3. And then, we can obtain a filtration $F \to S^3 \to K(\mathbb{Z}, 3)$. F is 3-connected and $\pi_i(F) \cong \pi_i(S^3)$ according to the long exact sequence of this fibration. Stretch $F \to S^3$ to another fibration $K(\mathbb{Z}, 2) \to X \to S^3$ where $X \simeq F$. The spectral sequence for this fibration have $E_2 = E_3$ pages shown in the figure.

E_2 page for $H^*(X)$ E_3 page for $H^*(X)$



We want to determin $H^*(X;\mathbb{Z})$. Since X is 3-connected, $H_2(X) = H_3(X) = 0$ by Hurewicz theorem, thus by universal coefficient theorem, $H^3(X) = 0$. That is $E_\infty^{0,3} = E_4^{0,3}$ must be zero, therefore the differential $\mathbb{Z}a \to \mathbb{Z}x$ is isomorphic. Hence $d_3 a = x$, which implies that $d_3(a^n) = na^{n-1}x$. From this we deduce that $H^i(X) = \mathbb{Z}_n$ when $i = 2n + 1$ and 0 when $i = 2n > 0$. The corresponding homology is that $H_i(X) = \mathbb{Z}_n$ when $i = 2n$ and 0 when $i = 2n - 1$.

Let $\mathfrak{S}$, *the Serre classes*, be one of three following classes:

1. $\mathcal{FG}$, finitely generated abelian groups.
2. $\mathcal{T}_p$, torsion abelian groups. And the order of each element is only divisible by numbers from a set P of primes.

3. $\mathcal{F}_p$, finite groups in $\mathcal{T}_p$.

Theorem 3.7: If X is a path-connected and $\pi_1(X)$ acts trivially on $\pi_n(X)$, then $\pi_n(X) \in \mathfrak{S}$ for all n iff $H_n(X;\mathbb{Z}) \in \mathfrak{S}$ for all $n > 0$.

Especially, homotopy groups of a simply connected space are all finitely generated if and only if all its homology groups are finitely generated. Hence $\pi_i(S^n)$ is finitely generated. The preceding theorem can be deduced by a more generally Hurewicz theorem:

Theorem 3.8: If X is path connected and π_1 acts trivially on π_i for all i. Suppose $\pi_i(X) \in \mathfrak{S}$ for $i < n$, the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X)$ is isomorphic up to mod $\mathfrak{S}$.

According to these two theorem, we obtain that the first p-torsion in $\pi_*(X) \cong \pi_*(S^3)$ is $\mathbb{Z}_p$ in π_{2p}. Let $p = 2$ we have $\pi_4(S^3) = \mathbb{Z}_2$.

In the last part of this section, the conclusion that π_i^s is finite for $i > 0$.

Theorem 3.9: $\pi_i(S^n)$ is finite when $i > n$, except for $\pi_{4k-1}(S^{2k})$. It is indeed a finite group taking the direct sum with $\mathbb{Z}$.

Proof: Since we already known that $\pi_i(S^1) = 0$ when $i > 1$, we can assume that $n > 1$. Then the condition for Serre spectral sequence qualified.

As usual, there is a map $S^n \to K(\mathbb{Z}, n)$ which induces an isomorphism between them. Get a fibration with fiber F. Then F is n-connected according to the long exact sequence, and $\pi_i(F) \cong \pi_i(S^n)$ for $i > n$. Stretch another fibration $K(\mathbb{Z}, n-1) \to X \to S^n$, with $X \simeq F$ from the map $F \to S^n$. Then we can apply the cohomological Serre spectral sequence with $\mathbb{Q}$ coefficients. When n is odd, E_n page is the following picture. For the same reason

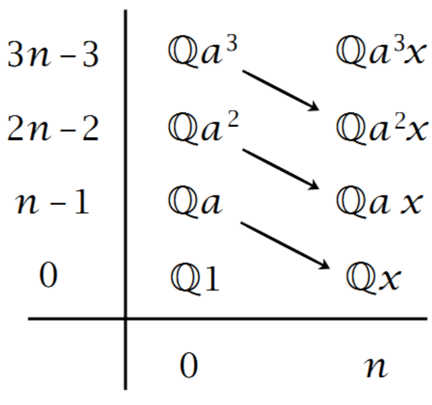


Figure 3-3 [7]

as previous examples, the differentials $\mathbb{Q}a \to \mathbb{Q}x$ must be isomorphism. Otherwise, it contradicts to X is $(n-1)$−connected. Hence all differentials are isomorphisms, which leads to $H^*(X;\mathbb{Q}) = 0 = H_*(X;\mathbb{Q})$, therefore $\pi_i(X) = \pi_i(S^n)$ is finite for all $i > n$

according to theorem 3.8.

When n is even, there are only the first two rows being nonzero in the picture. Since the cohomology of X is the same as S^{2n-1} in $\mathbb{Q}$ coefficients. Hence by the preceding theorem, $\pi_i(S^n)$ is finite when $n < i < 2n-1$ and a $\mathbb{Z}$ direct sum with a finite group through factoring out finite groups. When $i > 2n-1$, let Y be obtained from X by attaching cells of dimension greater than $2n-1$ such that $\pi_i(Y) = 0$ for $i \geqslant 2n-1$. Consider $X \hookrightarrow Y$ as a fibration, whose fiber is Z. Then Z is $(2n-2)-$connected according to the long exact sequence and $\pi_i(Z) \cong \pi_i(X)$ for $i \geqslant 2n-1$, and $\pi_i(Y) \cong \pi_i(X)$ for $i < 2n-1$ since adding cells greater than $2n-1$ does not affect π_i with $i < 2n-1$. Therefore $\pi_i(Y)$ is finite for all i which leads to $\tilde{H}^*(Y;\mathbb{Q})$ is finite and hence zero. Applying spectral sequence on $Z \to X \to Y$ yields $H^*(Z;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \cong H^*(S^{2n-1};\mathbb{Q})$. Then choose a map from Z to $K(\mathbb{Z}, 2n-1)$ inducing isomorphism on π_{2n-1}. Using the argument above, we have $\pi_i(Z)$ is finite for all i. Eventually, when $i > 2n-1$, $\pi_i(Z) \cong \pi_i(X)$ suggests $\pi_i(S^n)$ is finite for $i > 2n-1$. ∎

CHAPTER 4 COHOMOLOGY OPERATIONS

The main purpose of this chapter is to introduce the Steenrod operations and Adams spectral sequences as we mentioned in chapter 2. Steenrod operations as a kind of cohomology operations, is not just servicing for Adams spectral sequence. I give an example that determine the upper bound of tangent vector fields on sphere.

4.1 General Cohomology Operations

Definition 4.1: A *cohomology operation* is a transformation $\Theta = \Theta_X : H^m(X;G) \to H^n(X;H)$, with *fixed* m, n, G and H, and fit into the diagram.

$$\begin{array}{ccc} H^m(Y;G) & \xrightarrow{\Theta_Y} & H^n(Y;H) \\ \downarrow f^* & & \downarrow f^* \\ H^m(X;G) & \xrightarrow{\Theta_X} & H^n(X;H) \end{array}$$

If we view $H^m(\bullet;G)$ and $H^n(\bullet;H)$ as functors from the category of topological spaces to the category of groups or more generally modules, then we may regard a cohomology operation as a natural transformation between these two functors.

Example 4.1:

1. With coefficients in a ring R, the transformation $H^m(X;R) \to H^{mp}(X;R)$, $\alpha \mapsto \alpha^p$, is a cohomology operation since $f^*(\alpha^p) = (f^*(\alpha))^p$.
2. Taking $R = \mathbb{Z}$, the previous example says that a cohomology operation need not to be a homomorphism. It can be just between sets.

Proposition 4.1: For fixed m, n, G and H there is a bijection between $\Theta : H^m(X;G) \to H^n(X;H)$, all cohomology operations and $H^n(K(G,m);H)$, explicitly $\Theta \mapsto \Theta(\iota)$ where $\iota \in H^m(K(G,m);G)$ is a fundamental class.

Proof: Let X, Y be CW complex, so we can identity $H^m(X;G)$ with $\langle X, K(G,m)\rangle$. If an elment $\alpha \in H^m(X;G)$ corresponds to a map $\phi : X \to K(G,m)$, so that $\phi^*(\iota) = \alpha$, then $\theta(\alpha) = \Theta(\phi^*(\iota)) = \phi^*(\Theta(\iota))$, hence Θ is uniquely determined by $\Theta(\iota)$ since ϕ^* is uniquely determined by the class or element in $H^m(G;G)$. This provides injectivity. In the case of surjectivity, suppose $\alpha \in H^n(K(G,m);H)$ representing a map $\theta : K(G,m) \to K(H,n)$, then θ induces $\langle X, K(G,m)\rangle \to \langle X, K(H,n)\rangle$, that is, $\Theta : H^m(X;G) \to H^n(X;H)$ and $H^n(K(G,m);H)$, with $\Theta(\iota) = \alpha$ ∎

Cohomology operations must satisfy $m \geqslant n$. Since $K(G,m)$ being $(m-1)-$connected and applying universal coefficient theorem, yields $H^m(K(G,n);H) = 0$ when $m < n$. Moreover, since $H^m(K(G,m);H) \cong \mathrm{Hom}(G,H)$, cohomology operations fixing dimension are coefficient homomorphism. This proposition is analogue, in some sense, to the Yoneda Lemma in category theory.

4.2 Steenrod Operations

The interesting cohomology operations are *Steenrod squares* and *Steenrod powers* since they actually are homomorphisms:

$$Sq^i : H^n(X'\mathbb{Z}_2) \to H^{n+i}(X,\mathbb{Z}_2)$$
$$P^i : H^n(X;\mathbb{Z}_p) \to H^{n+2i(p-1)}(X;\mathbb{Z}_p) \text{ for odd primes p}$$

The Steenrod squares[8] $Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$, $i \geqslant 0$, satisfy the following properties.

1. $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f : X \to Y$, the naturality.
2. $Sq^i(\alpha + \beta) = Sq^i(\alpha) + Sq^i(\beta)$, being homomorphism.
3. $Sq^i(\alpha \smile \beta) = \sum_j Sq^j(\alpha) \smile Sq^{i-j}(\beta)$ (the Cartan formula[9]).
4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}_2) \to H^{n+1}(\Sigma X;\mathbb{Z}_2)$ is the suspension isomorphism.
5. $Sq^i(\alpha) = \alpha^2$ if $|\alpha| = i$, and $Sq^i(\alpha) = 0$ if $i > |\alpha|$.
6. $Sq^0 = \mathbb{1}$, the identity.
7. Sq^1 is the $\mathbb{Z}_2$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Recall the *Bockstein homomorphism.* If one has an exact sequence $0 \to A \to B \to C \to 0$ on abelian groups, then one can apply the covariant functor $\mathrm{Hom}(C_n(X);-)$ to yield an exact sequence $0 \to C^n(X;A) \to C^n(X;B) \to C^n(X;C) \to 0$. Thus we have:

$$\cdots \to H^n(X;A) \to H^n(X;B) \to H^n(X;C) \to H^{n+1}(X;A) \to \cdots$$

whose boundary map $\beta : H^n(X;C) \to H^{n+1}(X;A)$ is the *Bockstein homomorphism*.

Passing to Steenrod powers[10] $P^i : H^n(X;\mathbb{Z}_p) \to H^{n+2i(p-1)}(X;\mathbb{Z}_p)$, the similar properties holds:

1. $P^i f^* = f^* P^i$ for $f : X \to Y$, the naturality.
2. $P^i(\alpha + \beta) = P^i(\alpha) + P^i(\beta)$, being homomorphism.
3. $P^i(\alpha \smile \beta) = \sum_j P^j(\alpha) \smile P^{i-j}(\beta)$, the Cartan formula.[9]

4. Stable under suspension.
5. $P^i(\alpha) = \alpha^p$ if $2i = |\alpha|$, and $P^i(\alpha) = 0$ if $2i > |\alpha|$.
6. $P^0 = \mathbb{1}$.

It is clear that Steenrod squares and Steenrod powers are homomorphisms from (2). (4) means that the Sq^i's are stable under suspension. Steenrod firstly introduced Steenrod squares, with composition as product, these squares form an algebra, denoted $\mathcal{A}_2$, called the *mod 2 Steenrod Algebra*. The analogous Steenrod powers were constructed latter, and form the *mod p Steenrod Algebra*, $\mathcal{A}_p$. In the early 1950's, Cartan and Adem explored the structure of them. Then Serre and Cartan showed that Steenrod's constructions established all possible stable cohomology operations over the finite fields. The explicit constructions of Steenrod squares and powers are not important, instead, their strong properties are what we want.

The *total* Steenrod squares and powers are $Sq = Sq^0 + Sq^1 + \cdots$ and $P = P^0 + P^1 + \cdots$. Their action on $\alpha \in H^*(X, \mathbb{Z}_p)$ can only have finite Sq^i's or P^i's being nonzero . $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ and $P(\alpha \smile \beta) = P(\alpha)\sigma P(\beta)$ according to Cartan formulas, so that Sq and P are actually ring homomorphisms. Notice that $Sq(\alpha^n) = (Sq(\alpha))^n = (\alpha + \alpha^2)^n = \sum_i \binom{n}{i}\alpha^{n+i}$ when $|\alpha| = 1$, therefore $Sq^i(\alpha^n) = \binom{n}{i}\alpha^{n+i}$ when $|\alpha| = 1$.

Example 4.2 (Vector Fields on Spheres[5,11]): We can use Steenrod squares to find a upper bound of the number of independent tangent fields on spheres.

Recall the Stiefel manifold $V_{n,k}$ is a space whose points are orthonormal $k-$tuples in $\mathbb{R}^n$. Projecting a $k-$tuple onto its first coordinate is actually a map $p : V_{n,k} \to S^{n-1}$ with fiber $V_{n-1,k-1}$. A section corresponds to a set of $k-1$ orthonormal tangent vector fields on S^{n-1}.

The $(n-1)-$skeleton of $V_{n,k}$ is $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}$ for $2k-1 \leqslant n$. Now suppose we have $f : S^{n-1} \to V_{n,k}$ is a section. Since $pf = \mathbb{1}$, f^* is surjective on $H^{n-1}(-;\mathbb{Z}_2)$. By cellular approxiamation, we can assume f is cellular, that is $f : S^{n-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}$ if $2k-1 \leqslant n$. Hence $f*$ is an isomorphism due to the cellular cohomology of $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}$. If the number k satisfying $\binom{n-k}{k-1} = 1 \mod 2$, then the Steenrod square Sq^{k-1}:

$$H^{n-k}(\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1};\mathbb{Z}_2) \longrightarrow H^{n-1}(\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1};\mathbb{Z}_2)$$

$$\alpha^{n-k} \mapsto \binom{n-k}{k-1}\alpha^{n-1}$$

should be nontrivial. But f^* inducing an isomorphism, it contradicts to

$$Sq^{k-1} : H^{n-k}(S^{n-1};\mathbb{Z}_2) \to H^{n-1}(S^{n-1};\mathbb{Z}_2)$$

is zero.

Thus let $n = 2^r(2s+1)$ and $k = 2^r + 1$ with $s \geqslant 1$, then

$$\binom{n-k}{k-1} = \binom{2^{r+1}s-1}{2^r},$$

and in $\mod 2$ case is nonzero. The condition that $s \geqslant 1$ guarantees the condition $2k-1 \leqslant n$.

To conclude, for $n = 2^r(2s+1)$, the sphere S^{n-1} cannot have 2^r tangent fields if $s \geqslant 1$. When $s = 0$ this also holds, since S^{n-1} can not have n orthonormal tangent vector fields.

When $r \leqslant 3$, this result is optimal. Let $n = 2^r m$. When $r = 1$, there is only $2^1 - 1$ candidate. View S^{2m-1} as the unit sphere in $\mathbb{C}^m$, the unique tangent filed is $x \mapsto ix$. When $r = 2$, view S^{4m-1} as the unit sphere in $\mathbb{H}^m$, the maps $x \mapsto ix, jx, kx$ yields all tangent fields. As for $r = 3$, one performs the same procedure on octonions, $\mathbb{O}$. The upper bound is not best. The optimal one is obtained by K-theory, using the Adams operations.

Steenrod squares and powers can be composed, with quite complicated rules, called *Adem relations*[12]:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b$$

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j \quad \text{if } a < pb$$

$$P^a \beta P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j - \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \quad \text{if } a \leqslant pb$$

where the coefficients are taking in $\mathbb{Z}_2$. By convention, the binomial $\binom{m}{n}$ is zero if m or n is negative or if $m < n$. Though complicated at the first glance, it is still helpful in simplifying computation. For example, $Sq^1 Sq^b = (b-1)Sq^{b+1}$ so $Sq^1 Sq^{2i} = Sq^{2i+1}$ and $Sq^1 Sq^{2i+1} = 0$.

The mod 2 *Steenrod algebra* $\mathcal{A}_2$ is the algebra over $\mathbb{Z}_2$ generated by $Sq^1, Sq^2, \cdots$ quotient the ideal generated by the Adem relations. Similar to $\mathcal{A}_2$, $\mathcal{A}_p$ for odd prime p is defined to be the algebra over $\mathbb{Z}_p$ generated by $\beta, P^1, P^2, \cdots$ quotient the ideal generated by Adem relations and $\beta^2 = 0$. Thus $H^*(X, \mathbb{Z}_p)$ can be extend to a module over $\mathcal{A}_p$ rather then just $\mathbb{Z}_p$. Clearly, $\mathcal{A}_p$ is a graded algebra with element of degree k being maps $H^n(X, \mathbb{Z}_p) \to H^{n+k}(X; \mathbb{Z}_p)$ up to Adem relations for all n.

Proposition 4.2: $\mathcal{A}_2$ is generated as an algebra by elements of degree 2^k since there is a relation $Sq^i = \sum_{0<j<i} a_j Sq^{i-j} Sq^j$. $\mathcal{A}_p$ is generated as an algebra by elements of degree p^k with relation $P^i = \sum_{0<j<i} a_j P^{i-j} P^j$ with $a_j \in \mathbb{Z}_p$.

Proof: This is a little trick. The argument for $p = 2$ and odd p is the same. Assume p is odd. Let $i = i_0 + i_1 p + \cdots + i_k p^k$ with $i_k \neq 0$. Let $b = p^k$ and $a = i - b$ so that $a < pb$ and $a, b > 0$ if i is not a power of p. If we can show that $\binom{(p-1)b-1}{a}$ the $j = 0$ term is nonzero, then the conclusion follows from the Adem relation. It is indeed the case. The $p-$adic expansion of $(p-1)b - 1 = p^{k+1} - 1 - p^k = (p-1)(1 + p + \cdots + p^{k-1} + (p-2)p^k$ and $a = i_0 + i_1 p + \cdots + (i_k - 1)p^k$. It follows that

$$\binom{(p-1)b-1}{a} = \binom{p-1}{i_0} \cdots \binom{p-1}{i_{k-1}} \binom{p-2}{i_k - 1},$$

is nonzero. ∎

An elemnt a of a graded algebra is *decomposable* if it can be writen as $\sum_i a_i b_i$ with a_i, b_i having degree lower than a. This proposition implies that most of Steenrod operations are decomposable.

Using the argument above, we can show that the only spaces X with its cohomology ring with $\mathbb{Z}$ coefficients a polynomial ring $\mathbb{Z}[x]$ must satisfy the dimension of x is 2 or 4, corresponding to $\mathbb{CP}^\infty$ and $\mathbb{HP}^\infty$.

Theorem 4.1: Suppose $H^*(X; \mathbb{Z}_p)$ is the polynomial algebra on a generator α with $|\alpha| = n$. If $p = 2$, $n = 2^x$; if p is odd, then $n = p^k l$ where l divides $2(p-1)$ and is even.

Proof: When $p = 2$, $Sq^n(\alpha) = \alpha^2 \neq 0$ according to our hypothesis. If n is not a power of 2, then Sq^n decomposes into some $Sq^{n-i} Sq^i$ with $0 < i < n$. But such term must be zero since Sq^i maps anything into $H^{n+i}(X; \mathbb{Z}_2 = 0$ since $j < n$ and $H^*(X; \mathbb{Z}_2) = \mathbb{Z}[\alpha]$ with $|\alpha| = n$.

When p is odd, $\alpha^2 \neq 0$ implies that n is even. Suppose $n = 2k$, then $P^k(\alpha) = \alpha^p \neq 0$. Since P^k can be written as some P^{p^i}s, some $P^{p^i} \neq 0$ in $H^*(X; \mathbb{Z}_p)$. This implies n divides $2p^i(p-1)$. The result follows. ∎

Now if $H^*(X) = \mathbb{Z}[\alpha]$, passing from $\mathbb{Z}$ to $\mathbb{Z}_2$, the theorem yields that $|\alpha|$ is a power of 2. Passing to $\mathbb{Z}_3$, $|\alpha|$ is a power of 3 times a divisor of $2(3-1) = 4$. Hence $|\alpha| = 2, 4$.

4.3 Adams Spectral Sequences

We finally arrive here. In this section, I will establish the progress of calculation with Adams spectral sequences and calculate π_3^s as a special case of the motivated problem.

Theorem 4.2: [13] Let X, Y be CW complexes of finite type. And Y has finitely many cells. Then there is a spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_p}(\tilde{H}^*(Y, \mathbb{Z}_p), \tilde{H}^*(S^t X, \mathbb{Z}_p)) \Rightarrow \{S^{t-s}X, Y\}^\wedge_p.$$

This implies the statement in the last of chapter 2. The construction and proof of this theorem requires amounts of work. Using CW spectra and cofibration sequence for pair (X, A). Hence I will not present it here. Spectral sequences are machinery, it is enough for one using them in spite of understanding the constructions of them. To use Adams spectral sequence doing computation, one proceeds in three steps[14]

1. Calculate $\mathrm{Ext}^s_{\mathcal{A}_p}(\tilde{H}^*(Y, \mathbb{Z}_p), \tilde{H}^*(S^t X, \mathbb{Z}_p))$. This steps falls into two parts. Firstly, figure out the structure of $\tilde{H}^*(X; \mathbb{Z}_p)$ and $\tilde{H}^*(Y; \mathbb{Z}_p)$ as modules over $\mathcal{A}_p$. Secondly, calculate the Ext group using homological algebra. This is quite difficult.
2. Calculate E_{r+1} from E_r. This step is really hard, usually impossible for there are infinitely many tough terms to be computed.
3. Deduce $\{S^{t-s}X, Y\}^\wedge_2$ from E_∞ page. One obtains a filtration on it at most time.

Perhaps the greatest uses of the Adams spectral sequences are proving things rather than calculation. For instance, if a map $X \to X'$ induces isomorphism on E_2 pages, then inducing isomorphism on $\{X, Y\}^\wedge_2 \to \{X', Y\}^\wedge_2$.

As promised, let us compute the 2-component of π^s_*. The first step involves the calculation of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{Z}_2, \mathbb{Z}_2)$, where t means the latter $\mathbb{Z}_2$ viewed as a graded module, has the $\mathbb{Z}_2$ summand on the grading t. Homological algebra suggests us to do the deleted projective resolution for the former $\mathbb{Z}_2$ or deleted injective resolution for the latter, both viewed as graded modules with $\mathbb{Z}_2$ on the grading 0.

Generally, for computing $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X), \mathbb{Z}_p)$ it suffices to construct a minimal free resolution of $H^*(X)$ over $\mathcal{A}_p$:

$$\cdots \to F_2 \to F_1 \to F_0 \to H^*(X) \to 0$$

where at each step of the inductive construction, the number of generators of F_i chosen is minimal.

Proposition 4.3: For a minimal resolution, all maps in dual complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_2, \mathbb{Z}_p) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_1, \mathbb{Z}_p) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_0, \mathbb{Z}_p) \leftarrow 0$$

are zero, hence $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X), \mathbb{Z}_p) = \mathrm{Hom}^t_{\mathcal{A}_p}(F_s, \mathbb{Z}_p)$, t indicates the grading for $\mathbb{Z}_p$.

Proof: Denote $\mathcal{A}^+$ for the ideal in $\mathcal{A}$ generated by elements having degree nonzero. Observe that $\ker \phi_i : F_i \to F_{i-1} \subset \mathcal{A}^+ F_i$. Since if $x \in \ker \phi_i$ and $x = \sum_j a_j x_{ij}$ with

$a_j \in \mathcal{A}$ with some $a_j \in \mathcal{A}^0 = \mathbb{Z}_p$ nonzero, then we can solve the equation $0 = \phi_i(x) = \sum_j a_j \phi_i(x_{ij})$ for $\phi_i(x_{ij})$, which against to the minimal construction on F_i.

Since $\phi_{i-1}\phi_i = 0$, we have $\phi_i(x) \in \ker \phi_{i-1}$ for each $x \in F_i$ with $\phi_i(x) = \sum_j a_j x_{i-1,j}$ with $a_j \in \mathcal{A}^+$. Hence for each $f \in \mathrm{Hom}_{\mathcal{A}}(F_{i-1}, \mathbb{Z}_p)$ we have

$$\phi_i^* f(x) = f\phi_i(x) = \sum_j a_j f(x_{i-1,j}) = 0$$

since $a_j \in \mathcal{A}^+$ will send $f(x_{i-1,j}) \in \mathbb{Z}_p$ to zero. ∎

Return to the calculation on 2-component of π_*^s. Remind that we have already proved $\pi_1^s = \pi_4(S^3) = \mathbb{Z}_2$, since for $n > i+1$ $\pi_i^s = \pi_{n+i}(S^n)$. The E_2 page consists of terms $\mathrm{Hom}^t_{\mathcal{A}_2}(F_s, \mathbb{Z}_2)$ where F_s's being the minimal free resolution of $\mathbb{Z}_2$ in the category of graded $\mathcal{A}_2$-modules. Hence our task is to construct F_i's as we need.

Begin with $F_0 \to \mathbb{Z}_2$. Then F_0 must be a free $\mathcal{A}_2$ module generated by one generator in degree 0 denoted by ι with $\iota \mapsto \mathbb{1} \in \mathbb{Z}_2$. Hence $F_0 = \mathcal{A}_2\iota$, the first column of the table below. This map sends everything to zero except ι, thus the kernel is $\mathcal{A}_2^+$.

Then we consider $F_1 \to F_0$. Clearly, we need a α_1 with degree 1, which maps to $Sq^1\iota$. Therefore, there is a $\mathcal{A}_2\alpha_1 \subset F_1$. Notice that $Sq^1\alpha_1$ will map to $Sq^1Sq^1\iota$ in F_0, which is zero. We have no choice but introducing a new generator α_2 with degree 2 who maps to $Sq^2\iota$.

It is convenient to let Sq^I denote the composition $Sq^{i_1}Sq^{i_2}\cdots$. If no Adem relations can be applied to Sq^I, that is, $i_j \geqslant 2i_{j+1}$, then we say it *admissible*. By applying Adem relations iteratively, we can write every monomial Sq^I as a sum of admissible monomials .

Since α_1 maps to $Sq^1\iota$, it follows that $Sq^I\alpha_1$ is sent to $Sq^ISq^1\iota$ for all admissible I except those end with 1. In particular, $Sq^1\alpha_1$ maps to zero. And we need the α_2 as explained above. All generators we need in the second column are α_{2^n}'s which are mapped to $Sq^{2^n}\iota$ since Sq^{2^n} can not be decomposed. Also notice that F_i starts with a generator with degree i by induction, as one can see in the first two rows of the table below.

Subsequent columns are computed in this way. And by finding out all generators, we can compute the E_2 page. But this is unpractical, since one can never know whether he can move to the next column. Instead, one can compute the generators of a whole row by vanishing line theorem.

The calculation show that the portion of E_2 page in mod 2 case is in the following table 4.1. The horizontal coordinate stand for $t-s$ and the vertical for s. Changing the coordinates is convenient for investigating π_n^s, for all its factor groups lying on the column

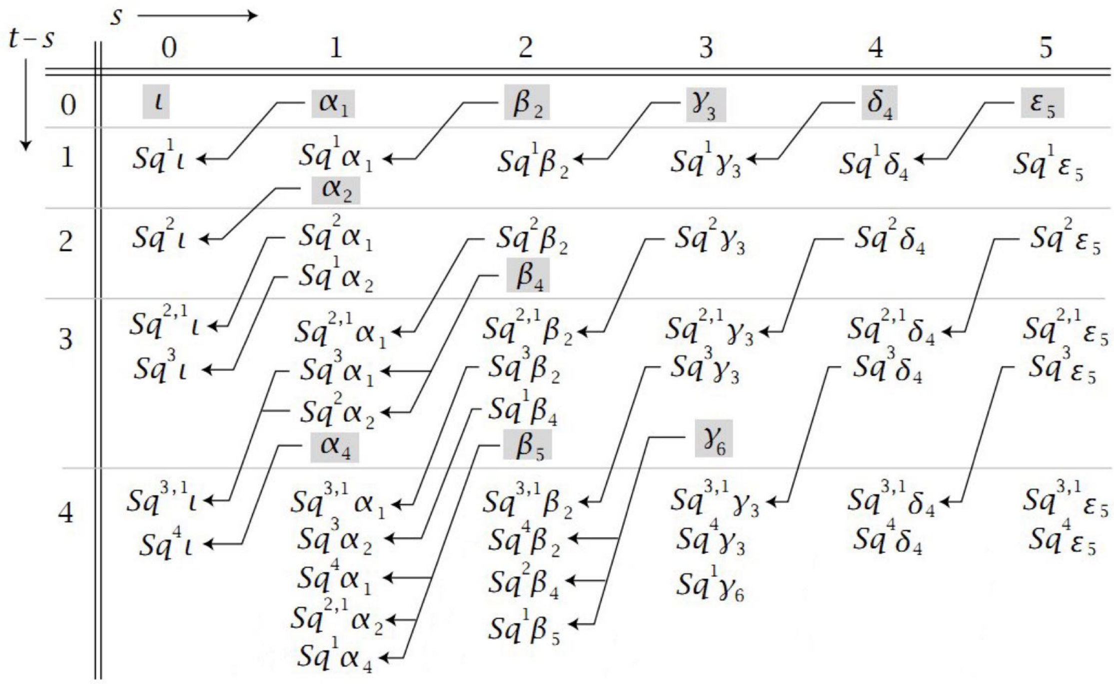
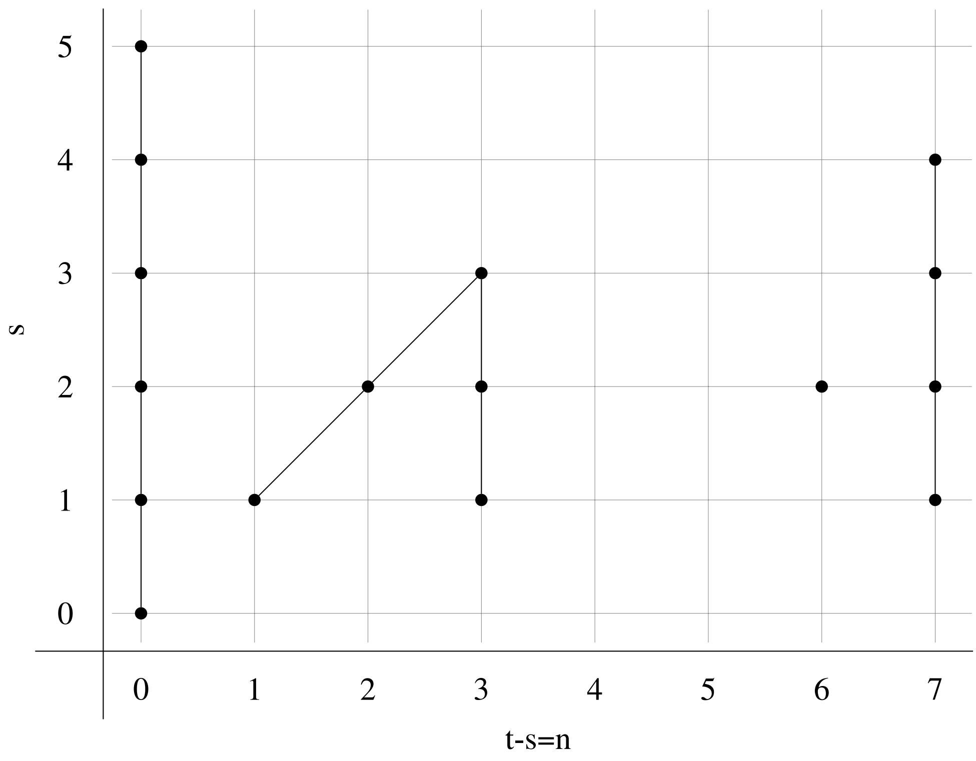


Figure 4-1 Resolution for $\mathbb{Z}_2$ [5]

$t - s = n$.

Mod 2 Adams spectral sequence for π_*^s



The second step is to consider the differentials d_2. In this coordinates change, differentials d_r has degree $(-1, r)$. Hence all differentials are zero except for $d_2^{1,2}$ starting from $(1, 2)$ to $(0, 4)$ in this page. Hence are terms are stable except the term at $(1, 2)$ position.

But we already know that $\pi_1^s = \pi_4(S^3) = \mathbb{Z}_2$, the differential $d_r^{1,2}$ must be zero for all $r \geqslant 2$. Then according to Serre spectral sequences, all stable homotopy groups π_i^s are finite except for $i = 0$, we conclude that the orders of ${}_{(2)}\pi_2^s$ and ${}_{(2)}\pi_3^s$ are 2 and 8 respectively. As for ${}_{(2)}\pi_3^s$, one still needs to determine the structure of this group with order 8. Actually, ${}_{(2)}\pi_*^s$ is a graded ring. There is a fact states that ${}_{(2)}\pi_3^s = \mathbb{Z}_8$ using the graded ring structure $\pi_i^s \times \pi_j^s \to \pi_{i+j}^s$ defined by compositions $S^{i+j+k} \to S^{j+k} \to S^k$. To clarify this explicitly requires more work, thus not included.

Calculations using Adams spectral sequence is really complicated due to the structure of Steenrod algebra. In Appendix, I included some charts of Adams spectral sequences in different prime p made by Hood Chatham. Greenlees generalized this approach for some other cohomology theory, Adams specral sequences yield some different information about $\{X, Y\}$, listing in the table below.

Table 4-1 Result of Greenlees

Cohomology theory	Information
$H^*(\cdot\,;\mathbb{Z}_2)$	$\{X, Y\}_p^\wedge$
$H^*(\cdot\,;\mathbb{Q})$	$\{X, Y\} \otimes \mathbb{Q}$
$H^*(\cdot\,;\mathbb{Z})$	$\{X, Y\}$
$K^*(\cdot)$	A periodic form of $\{X, Y\}$
$MU^*(\cdot)$	$\{X, Y\}$

$K^*(\cdot)$ stands for K-theory. The periodicity is an attenuated form of Bott periodicity and mixes up $\{S^k X, Y\}$ for various k. $MU^*(\cdot)$ is the complex cobordism, which requires a lot of hard work as preparation. When X, Y are both spheres, this method is the most efficient known method for calculation at odd primes.

4.4 K-Theory and Adams Operations

4.4.1 K-Theory

K-theory is the first generalized cohomology theory, invented by Atiyah and Hirzebruch around 1960[15], based on the Bott's Periodicity Theorem[16]. The idea is to consider all vector bundles over a space X forming an abelian group, in fact ring structure.

If E_1 and E_2 are vector bundles over B, we say they are stably isomorphic if $E_1 \oplus \epsilon^n \cong E_2 \oplus \epsilon^n$, denoted by $E_1 \sim_s E_2$, ϵ^n is the $n-$dimensional trivial bundle. This is an equivalence relation. Under this relation, we can define the addition $E_1 + E_2$ to be $E_1 \oplus E_2$, where E_i representing the equivalent class of its own. It is associative, commutative, with identity element ϵ^0. Bui int this setting, only ϵ^0 has its own as inverse. Hence we add the formal inverse $-E$ of E for all E, and $E_1 - E_1' = E_2 - E_2'$ if $E_1 + E_2' \sim_s E_2 + E_1'$. Denote this group by $K(X)$. Notice that any element in $K(X)$ can be written as $E - E'$ and the zero element is the class of $E - E$ for all E. Further, $E - E' = (E + E'^{\perp}) - (E' + E'^{\perp}) = (E + E'^{\perp}) - \epsilon^n$. Therefore the elements in $K(X)$ can be written as $E - \epsilon^n$.

Similarly, there is another equivalence relation. We say $E_1 \sim E_2$ if $E_1 \oplus \epsilon^m \cong E_2 \oplus \epsilon^n$ for some m and n. The set of all vector bundles over X under this relation then forms an abelian group automatically, since for any E, there is a vector bundle $E^{\perp}$ such that $E \oplus E^{\perp} \cong \epsilon^n$ for some n. Denote this group by $\tilde{K}(X)$.

There is a natural homomorphism from $K(X)$ to $\tilde{K}(X)$, namely sending the class of $E - \epsilon^n$ to the class of E. The kernel of this morphism is the class of $E - \epsilon^n$ with $E \cong \epsilon^m$ for some m, that is the subset $\{\epsilon^m - \epsilon^n\}$ of $K(X)$. In fact, restricting of E over X to a base point x_0 gives a homomorphsim from $K(X)$ to $K(x_0)$ which restricts on $\{\epsilon^m - \epsilon^n\}$ is an isomorphsim. Hence $K(X)$ splits as $K(X) = K(x_0) \oplus \tilde{K}(X) = \tilde{K}(X) \oplus \mathbb{Z}$.

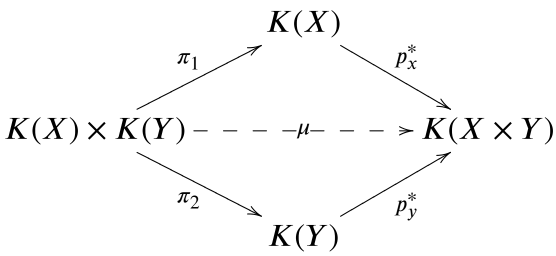
We can also define multiplication over $K(X)$, which is

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2'.$$

Multiplication over $K(X)$ is associative, commutative, and satisfies distribution. Hence $K(X)$ is a commutative ring with identity elements ϵ^0.

$K(\cdot)$ is a contravariant functor from the category of topological spaces to the category of commutative rings, satisfying if $f \simeq g : X \to Y$, then $f^* = g^* : K(Y) \to K(X)$. $\tilde{K}(X)$ identified with the kernel of $K(X) \to K(x_0)$ is clearly an ideal of $K(X)$ and a ring of its own.

Consider projections p_x and p_y from $X \times Y$ to X and Y. Applying the functor $K(\cdot)$, we get a commutative diagram



$\mu : (a, b) \mapsto p_x^*(a) p_y^*(b)$, which is bilinear. Therefore we get a morphism called *cross* *product* or *external product* as in ordinary cohomology theory:

$$\tilde{\mu} : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$
$$a \otimes b \mapsto a * b = p_x^*(a) p_y^*(b).$$

Let $Y = S^2$, we have a map $K(S^2) \otimes K(X) \to K(S^2 \times X)$. In $K(S^2)$, let H be the complex line bundle over $\mathbb{CP}^1 = S^2$, we have a relation $(H \otimes H) \oplus 1 = H \oplus H$, that is $H^2 + 1 = 2H$. Hence there is a map from the polynomial ring $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$, which is an isomorphism in fact. And the cross product $K(S^2) \otimes K(X) \to K(S^2 \times X)$ is an isomorphism as well.

Theorem 4.3 (The Fundamental Product Theorem[17-18]): The homomorphism μ : $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.

Taking X as a point, one finds $\tilde{K}(S^2)$ is generated by $H - 1$, and multiplication in $\tilde{K}(S^2)$ is trivial sicne $(H-1)^2 = 0$.

Now we can extend $\tilde{K}(\cdot)$ to a cohomology theory. Suppose we have $A \subset X$ and maps $A \overset{i}{\hookrightarrow} X \overset{j}{\to} X/A$ between topological spaces. Applying the functor $\tilde{K}(\cdot)$ yields $\tilde{K}(X/A) \overset{j^*}{\to} \tilde{K}(X) \overset{i^*}{\to} \tilde{K}(A)$. It is clear that the image of j^* contained in $\ker i^*$. For the opposite, suppose E over X is trivial when restricts on A. Choosing a trivialization $h : p^{-1}(A) \to A \times \mathbb{C}^n$. Construct E/h as the quotient space of E under the identifications $h^{-1}(x, v) \sim h^{-1}(y, v)$ for $x, y \in A$. Then E/h is a vector bundle over X/A and $j^*(E/h) = E$ in $\tilde{K}(X)$.

For pair (X, A), the *cofibration sequence* is

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \hookrightarrow,$$

By collapsing contractible subspaces, the above sequence can be written as

$$A \hookrightarrow X \hookrightarrow X/A \hookrightarrow SA \hookrightarrow SX \hookrightarrow S(X/A) \hookrightarrow \cdots$$

We have shown that $\tilde{K}(\cdot)$ is exact on cofibration sequence, hence there is an exact sequence,

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A).$$

If $X = A \vee B$, then $X/A = B$ and the sequence splits, $\tilde{K}(X) = \tilde{K}(A) \oplus \tilde{K}(B)$ since the last map is restriction on A which is surjective.

As cross product in $K(\cdot)$, we can define the corresponding cross product in $\tilde{K}(\cdot)$. For $a \in \tilde{K}(X) = \ker(K(X) \to K(x_0)$ and $b \in \tilde{K}(Y) = \ker(K(Y) \to K(y_0))$, then external product $a * b = p_1^*(a) p_2^*(b) \in K(X \times Y)$ has $p_1^*(a)$ restricting zero in $K(Y)$ and $p_2^*(b)$

restricting zero in $K(X)$. So $p_1^*(a)p_2^*(b)$ restricts to zero in both $K(X)$ and $K(Y)$, hence on $K(X \vee Y)$. This defines the reduced cross product $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \times Y, X \vee Y) = \tilde{K}(X \wedge Y)$.

Since $S^n \wedge X = \Sigma^n X$ which is homotopic equivalent to $S^n X$. Taking $Y = S^2$, we have a map:

$$\beta : \tilde{K}(X) \to \tilde{K}(S^2 X), \quad \beta(a) = (H - 1) * a$$

where H is the canonical line bundle over $S^2 = \mathbb{CP}^1$.

Theorem 4.4 (Bott Periodicity Theorem[18]): The homomorphism $\beta : \tilde{K}(X) \to \tilde{K}(S^2 X)$, $\beta(a) = (H - 1) * a$ is an isomorphsim for all compact Hausdorff spaces X.

Proof: The external product on $\tilde{K}(\cdot)$ is actually the restriction of external product in $K(\cdot)$ on $\tilde{K}(\cdot)$. Since the fundamental product theorem is an isomorphism, the consequence follow. ∎

Hence by Bott Periodicity Theorem, $\tilde{K}(S^{2n+1}) = \tilde{K}(S^1) = 0$; $\tilde{K}(S^{2n}) = \mathbb{Z}$, the generator is $(H - 1) * \cdots * (H - 1)$. The first statement comes from all complex bundles over S^1 are trivial.

If we set $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ and $\tilde{K}^{-n}(X, A) = \tilde{K}(S^n(X/A))$, the sequence above can be rewritten into:

$$\tilde{K}^{-2}(X) \to \tilde{K}^{-2}(A) \to \tilde{K}^{-1}(X, A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^{0}(X, A) \to \tilde{K}^{0}(X) \to \tilde{K}^{0}(A)$$

Define $\tilde{K}^{2i}(X) = \tilde{K}(X)$ and $\tilde{K}^{2i+1}(X) = \tilde{K}(SX)$ for $i \geqslant 0$. The long exact sequence can summarized as following periodic diagram.

$$\begin{array}{ccccc} \tilde{K}^0(X, A) & \longrightarrow & \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(A) \\ \uparrow & & & & \downarrow \\ \tilde{K}^1(A) & \longleftarrow & \tilde{K}^1(X) & \longleftarrow & \tilde{K}^1(X, A) \end{array}$$

A product $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y)$ can be defined as previous in the obvious way. Let $\tilde{K}^*(X) = \tilde{K}^0(X) \oplus \tilde{K}^1(X)$, then this gives a product $\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to \tilde{K}^*(X \wedge Y)$. The relative form of this is a product $\tilde{K}^*(X, A) \otimes \tilde{K}^*(Y, B) \to \tilde{K}^*(X \times Y, X \times B \cup A \times Y)$.

If we compose the external product $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \to \tilde{K}^*(X \wedge X)$ with the diagonal map $X \to X \times X$, then we have a multiplication on $\tilde{K}^*(X)$ making it into a ring, and extending the previously ring structure on $\tilde{K}^0(X)$.

Proposition 4.4: The multiplication is graded commutative, $\alpha\beta = (-1)^{ij}\beta\alpha$ for $\alpha \in \tilde{K}^i(X)$ and $\beta \in \tilde{K}^j(X)$.

4.4.2 Adams Operations and Division Algebras

In this section, we will use Adams operations to prove the celebrating theorem of Adams which asserts that:

Theorem 4.5: [19] There exists a map $f : S^{4n-1} \to S^{2n}$ of Hopf invariant ± 1 only when $n = 1, 2, 4$.

This theorem has famous applications, for instance, $\mathbb{R}^n$ is a division algebra and S^{n-1} is parallelizable only for $n = 1, 2, 4, 8$. Of course, we can deduce that $\mathbb{R}^n$ has a division algebra structure only when n is a power of 2. But using K-theory, the conclusion is more powerful. Starting with the definition of H-space.

In our case, an H-space structure on S^{n-1} is a continuous map $S^{n-1} \times S^{n-1} \to S^{n-1}$ which has identity element $e \in S^{n-1}$. We do not assume the existence of inverses and associativity of the multiplication.

Proposition 4.5: If $\mathbb{R}^n$ is a division algebra, then S^{n-1} is an H-space.

Proof: This is because if we have a division algebra structure, then we can define the H-space structure on S^{n-1} by $(x, y) \mapsto xy/|xy|$. ∎

Since we have the isomorphism $\tilde{K}(S^{2k}) \otimes \tilde{K}(X) \to \tilde{K}(S^{2k} \wedge X)$, the external product on $K(S^{2k}) \otimes K(X) \to K(S^{2k} \times X)$ is also an isomorphism. And $K(S^{2k})$ can be described as $\mathbb{Z}[\alpha]/(\alpha^2)$, we can deduce that $K(S^{2k} \times S^{2l})$ is $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. An additive basis for $K(S^{2k} \times S^{2l})$ is $\{1, \alpha, \beta, \alpha\beta\}$.

We can from this deduce that S^{2k} is not an H-space for $k > 0$. Suppose $\mu : S^{2k} \times S^{2k} \to S^{2k}$ is an H-space multiplication. It induces homomorphism $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. The composition $S^{2k} \overset{i}{\hookrightarrow} S^{2k} \times S^{2k} \overset{\mu}{\to} S^{2k}$ is the identity, where i is the inclusion onto either of the subspaces $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$. The map i^* induced by inclusion into the first factor sends α to γ and β to 0, hence the coefficient of α in $\mu^*(\gamma)$ must be 1, similarly, coefficient for β is 1 and no constant term. Therefore $\mu^*(\gamma) = \alpha + \beta + t\alpha\beta$. Now using $K(S^{2k})$ has trivial ring structure and μ^* is a ring homomorphism,

$$0 = \mu^*(\gamma^2) = (\alpha + \beta + t\alpha\beta)^2 = 2\alpha\beta \neq 0,$$

a contradiction. Hence $\mathbb{R}^n$ may have a division algebra structure only when n is even.

It remains to show that S^{n-1} is not a H-space when n is even and different from 2,4,8. This needs the concept of Hopf invariant.

If $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ is a map, we can associate a map $\hat{g} : S^{2n-1} \to S^n$, defined as follow. Regard S^{2n-1} as $\partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n$, and S^n the union of D^n_+ and D^n_-. Then $\hat{g}$ is defined on $\partial D^n \times D^n$ by $\hat{g}(x, y) = |y|g(x, y/|y|) \in D^n_+$ and on $D^n \times \partial D^n$ by $\hat{g}(x, y) = |x|g(x/|x|, y) \in D^n_-$. $\hat{g}$ agrees with g on $S^{n-1} \times S^{n-1}$.

Now focus on n is even, so replace n by $2n$. For a map $f : S^{4n-1} \to S^{2n}$, let C_f be S^{2n} with a cell e^{4n} attached by f, thus C_f/S^{2n} is S^{4n}. There is a short exact sequence

$$0 \to \tilde{K}(S^{4n}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0.$$

Let $\alpha \in \tilde{K}(C_f)$ be the image of the generator of $\tilde{K}(S^{4n})$ and $\beta \in \tilde{K}(C_f)$ map to the generator of $\tilde{K}(S^{2n})$. It follows from β^2 maps to 0, hence $\beta^2 = h\alpha$. The *Hopf invariant* of f is defined to be the integer h.

Proposition 4.6: If g is an H-space multiplication on S^{2n-1}, then the associated map $\hat{g} : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1.

The result then follows from the Adams Hopf invariant one theorem stated at the beginning.

To show the theorem of Adams, we need more structure on $K(X)$. Just as Steenrod operations on cohomology rings, we have Adams operations on the K rings.

Theorem 4.6: [20] There are ring morphisms $\Psi^k : K(X) \to K(X)$, for any $k \geqslant 0$ satisfying:

1. $\Psi^k f^* = f^* \Psi^k$ for all maps $f : X \to Y$.
2. $\Psi^k(L) = L^k$ if L is a line bundle.
3. $\Psi^k \circ \Psi^l = \Psi^{kl}$.
4. $\Psi^p(\alpha) = \alpha^p \bmod p$ for p prime.

X need to be compact Hausdorff spaces.

As Steenrod operations, we do not need the explicit construction of Adams operations at most time. Their properties can help us a lot. For example, $\Psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ must be a multiplication by an integer for $\tilde{K}(S^{2n}) = \mathbb{Z}$. In the case $n = 1$, let $\alpha = H - 1$ be the generator of $\tilde{K}(S^{2n})$. Then

$$\begin{aligned}\Psi^k(\alpha) &= \Psi^k(H-1) = H^k - 1 \\ &= (1+\alpha)^k - 1 \\ &= 1 + k\alpha - 1 \\ &= k\alpha\end{aligned}$$

And hence $\Psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k^n by induction.

Finally, we can start to prove the Adams theorem on Hopf invariant. Recall that $\alpha \in \tilde{K}(C_f)$ is the image of the generator of $\tilde{K}(S^{4n})$ and $\beta \in \tilde{K}(C_f)$ mapped to the generator of $\tilde{K}(S^{2n})$ with the relation $\beta^2 = \pm\alpha$. Now $\Psi^k(\alpha) = k^{2n}\alpha$ and $\Psi^k(\beta) = k^n\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$. Therefore

$$\Psi^k\Psi^l(\beta) = \Psi^k(l^n\beta + \mu_l\alpha) = k^n l^n\beta + (k^{2n}\mu_l + l^n\mu_k)\alpha.$$

Since $\Psi^k\Psi^l = \Psi^{kl} = \Psi^l\Psi^k$, we must have the relation

$$k^{2n}\mu_l + l^n\mu_k = l^{2n}\mu_k + k^n\mu_l$$

or equivalently,

$$(k^{2n} - k^n)\mu_l = (l^{2n} - l^n)\mu_k.$$

By letting $k = 2$, we have $\Psi^2(\beta) = \beta^2 \bmod 2$. Since $\beta^2 = h\alpha$ with $h = \pm 1$ the Hopf invariant, the formula $\Psi^2(\beta) = 2^n\beta + \mu_2\alpha$ implies $\mu_2 = h \bmod 2$, so μ_2 must be odd. By letting $k = 3$, we have $2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2$. Hence 2^n divides $3^n - 1$ for 3^n and μ_2 both odd. But 2^n divides $3^n - 1$ only holds for $n = 1, 2, 4$ by elementary number theory fact.

Therefore $\mathbb{R}^n$ has division algebra structure only for $n = 1, 2, 4, 8$ corresponding to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The first two have indentities and the last one dose not satisfy associativity.

CONCLUSION

Computing stable homotopy groups is a core work in algebraic topology. And now we can at least tell something about stable maps $\{X, Y\}$ between X and Y, that is, $\{X, Y\}$ with its $p-$component. Though we have powerful tools such as, Adams spectral sequences, the complicated and tedious work often makes one confusing. To avoid or at least ensure the method is effective, we need to analyze our problem as further as possible, instead of throwing all stuff into the machinery. The spectral sequences method can also be used in some new cohomology theory, such as topological modular forms, which is one aspect in my Ph.D. study. In spite of this, cohomology operations show its own strength by deducing an upper bound of the number of independent vector fields over S^{n-1} is $2^r - 1$ when $n = 2^r(2s + 1)$ and proving that there is no space with polynomial ring as its cohomology ring except $\mathbb{CP}^\infty$ and $\mathbb{HP}^\infty$. The last two section show the power of K-theory, a generalized cohomology theory through the classical result on Hopf invariant one and division algebra structures over $\mathbb{R}$, which indicates that new homology or cohomology theories is needed. This master degree thesis emphasizes the absurdity of ignoring the calculational aspect of theories and methods aiming to the philosophy of algebraic topology.

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APPENDIX A

A.1 Mod 2 Adams spectral sequence

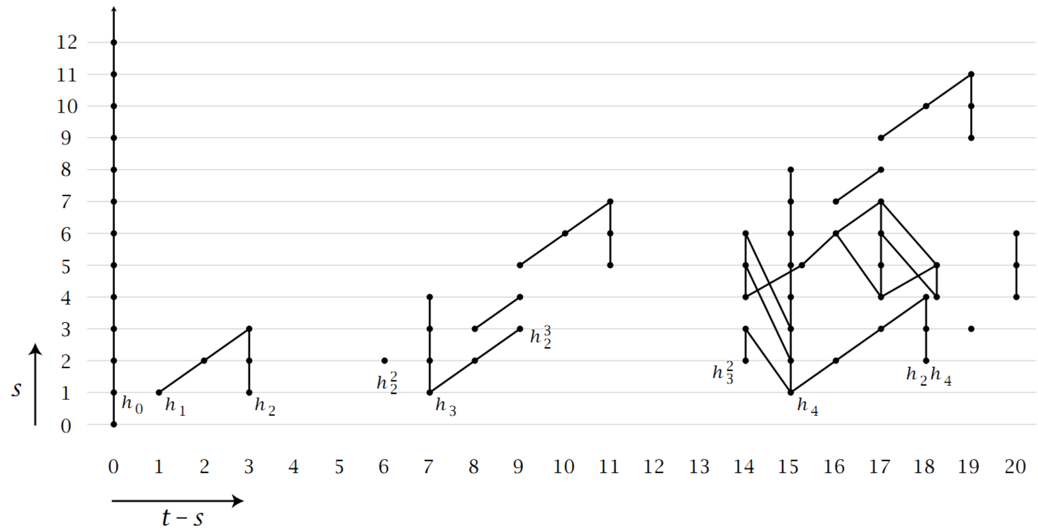

Figure A-1 Mod 2 Adams spectral sequence for π_*^s [5]

This figure from[5] shows the 2-component of π_n^s up to $n \leqslant 20$. The structure lines displayed in this figure mean there are product structures between them.

A.2 Calculation on mod 2 Steenrod Algebra

Recall that if $a < 2b$, then

$$Sq^a Sq^b = \sum_{j \geqslant 0} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j.$$

Applying this, when $a = 1$, one obtains that $Sq^1 Sq^b = (b-1)Sq^{b+1}$. Hence $Sq^1 Sq^{2n} = Sq^{2n+1}$ and $Sq^1 Sq^{2n-1} = 0$.

This applies to $Sq^1 \alpha_2 = Sq^1 Sq^2 \iota = Sq^3 \iota$, $Sq^1 \beta_4 = Sq^1(Sq^3 \alpha_1 + Sq^2 \alpha_2) = Sq^1 Sq^2 \alpha_2 = Sq^3 \alpha_2$, and calculates all relation in the table established in section 4.3. And there are more things we can compute.

In the second column, $Sq^{2,1}\alpha_2$ maps to $Sq^2 Sq^1 Sq^2 \iota = Sq^2 Sq^3 \iota = Sq^5 + Sq^4 Sq^1$ and $Sq^1 \alpha_4$ maps to $Sq^1 Sq^4 \iota = Sq^5$ by identities above.

When $a = 2$, $Sq^2 Sq^b = \binom{n-1}{2} Sq^{n+2} + Sq^{n+1} Sq^1$. This can help in computing $Sq^2 \alpha_4$ maps to $\binom{3}{2} Sq^6 + Sq^5 Sq^1$ as shown in the figure.

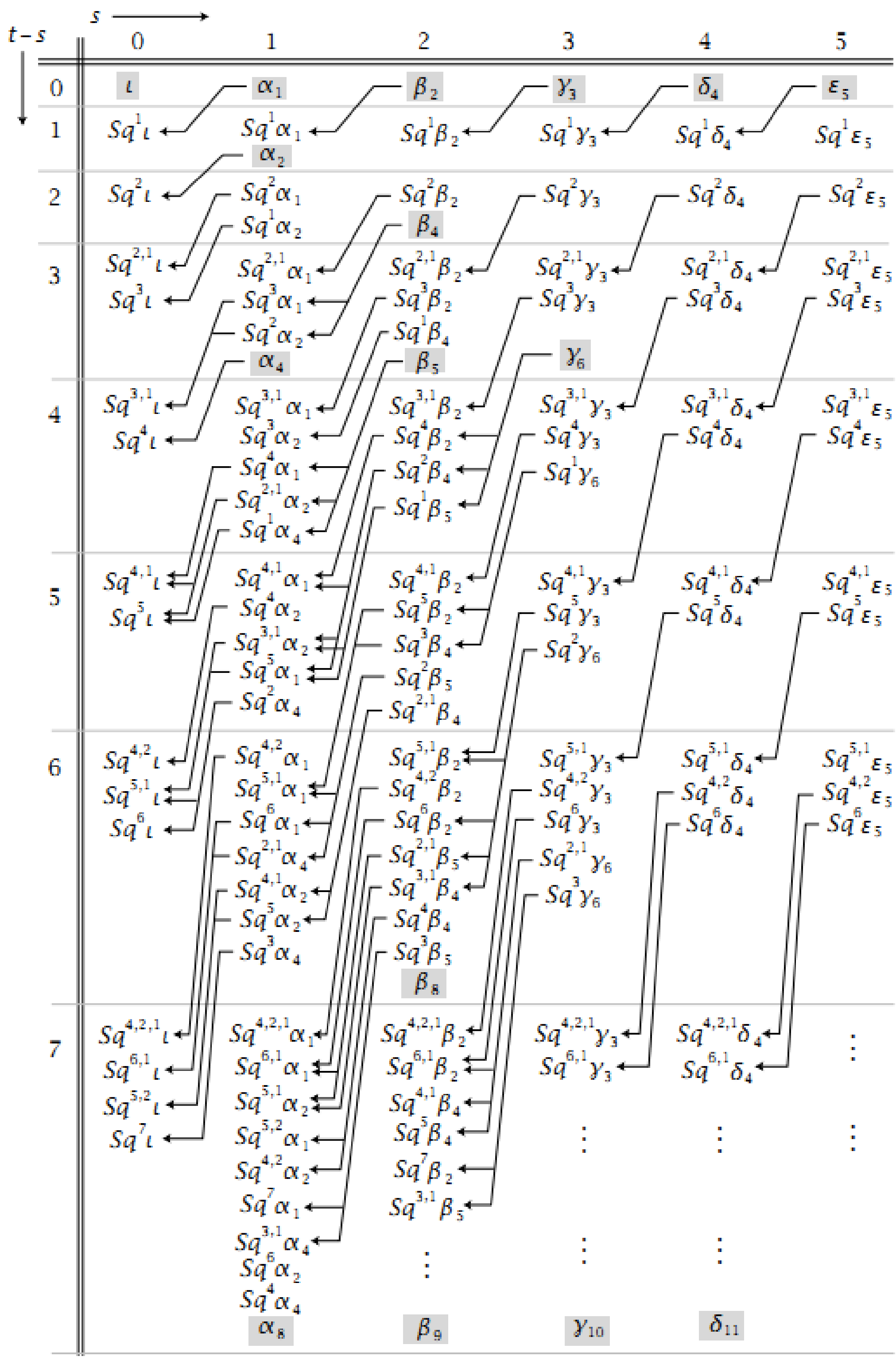

Figure A-2 Resolution for $\mathbb{Z}_2$ as $\mathcal{A}$-module[5]

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I hereby declare that this dissertation is my own original work under the guidance of my supervisor. It does not contain any research results that others have published or written. All sources I quoted in the dissertation are indicated in references or have been indicated or acknowledged. I shall bear the legal liabilities of the above statement.

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RESUME

Born in Anhui Province, Bengbu City, on January, 5th, 1997.

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