# Power Operations on K(n-1)-Localized Morava E-Theories of Height n

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# Contents

# FGLs

$R$: a commutative ring, $F \in R[\![X, Y]\!]$. If $F$ satisfies

- $F(X, 0) = X$ and $F(X, Y) = F(Y, X)$,
- $F(F(X, Y), Z) = F(X, F(Y, Z))$,

we say $F$ is a formal group law(FGL) over $R$.

Example: $F = X + Y \in \mathbb{Z}[\![X, Y]\!]$, additive
$F = X + Y + XY \in \mathbb{Z}[\![X, Y]\!]$, multiplicative

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Origin: Algebraic number theory$\Longrightarrow$ Application: Algebraic Topology

# From spectra to FGLs

Suppose $L$ and $L'$ are two complex line bundles over $X$ and $E$ is a multiplicative complex oriented cohomology theory.

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In the universal case, $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifies tensor product of line bundles, correspondingly

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Example: $E = H(-;\mathbb{Z})$, $F_E = X + Y$ is additive;
$E = KU$, the complex $K$-theory, $F_E = X + Y + XY$ is multiplicative.

# The universal FGL over the Lazard ring

Universal formal group law: $F = X + Y + \Sigma a_{ij} X^i Y^j$ over

$$L = \mathbb{Z}[a_{ij}]/\sim$$

Quillen 69's work [Qui07]: $L \cong \mathbb{Z}[x_1, x_2, \cdots] \cong MU^*$.

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Quillen 69's work [Qui07]: $L \cong \mathbb{Z}[x_1, x_2, \cdots] \cong MU^*$.

Corollary: If $E$ is a multiplicative complex oriented spectrum, then there is a unique map

$$MU \to E$$

between ring spectra, classifies the formal group $F_E$ over $E^*$.

# How to come back?

If we have a ring $R^*$ carrying a formal group law $F_R$, is there a spectrum $E_R$ with the formal group law $F_R$ over $\pi_{-*}E_R = R^*$?

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If we have a ring $R^*$ carrying a formal group law $F_R$, is there a spectrum $E_R$ with the formal group law $F_R$ over $\pi_{-*}E_R = R^*$?

Best choice: $E_{R*}(X) = MU_*(X) \otimes R^*$.

Landweber Exact Functor Theorem [Lan76]: $E_{R*}(-)$ is a homology theory if the images of $v_i = x_{p^i-1}$ ($v_0 = p$) in $R^*$ forms a regular sequence. i.e.

$$v_i : R/(p, v_0, \ldots, v_{i-1}) \hookrightarrow R/(p, v_0, \ldots, v_{i-1})$$

is an injection.

# Deformation of FGLs

Fix a perfect field $k$, $p = 0$ and $F$ a FGL over $k$.
$[p]_F(x) := \underbrace{x +_F x +_F \cdots +_F x}_{p} = \Sigma a_{p^n} x^{p^n} + \cdots$
This $n$ is called the height of $F$.

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Example: Over $\mathbb{F}_p$, $[p]_{F_a}(x) = 0$, has height 0;
$[p]_{F_m}(x) = x^p$, has height 1.

# Deformation of FGLs

A deformation of a height $n$ FGL $F$ over a perfect field $k$ consists of

- A FGL $\widetilde{F}$ over ring A, a complete local ring with residue field $k$ and
- $\pi_* \widetilde{F} = F$ over $k$, $\pi : A \to k$.

Example: The FGL $F_m = X + Y + XY$ over $\mathbb{Z}_p$ is a deformation of $F_m$ over $k$.

**Theorem (Lubin-Tate, 65, [LT65])**

*Deformations of $F$ over $k$ is classified by the ring*

$$A_0 := W(k)[[u_1, \ldots, u_{n-1}]].$$

# Morava E-theories

The ring $A_0$ satisfies the LEFT condition:

$$p \mapsto p, v_i \mapsto u_i$$

The resulting spectrum $E(k, F)$ is called a Morava $E-$theory, with

$$\pi_* E(k, F) = W(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm}]$$

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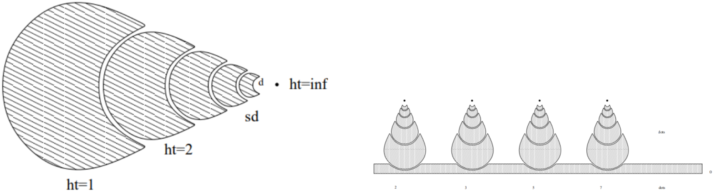
Remark: It can be shown that different choices of particular FGLs resulting homotopy equivalent spectra, thus we will denote $E(k, F)$ by $E_n$.
Remark: Hopkins, Goerss had further shown that the multiplicative structures over $E_n$ are essentially unique.

# Straightification of $\mathcal{M}_{fg}$




(a) (b)

Figure: Straightification of $\mathcal{M}_{fg}$, [DFHH14]

Chromatic Convergence Theorem:

$$X \cong \lim \cdots \to L_{E_2}X \to L_{E_1}X \to L_{E_0}X.$$

# Power operations

Suppose $E$ is a commutative ring spectrum/multiplicative cohomology theory.
$f : S \to E$ is an element in $E^0$.

$$f^m : S \to S \wedge \cdots \wedge S \to E \wedge \cdots \wedge E \xrightarrow{\mu} E$$

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$$P^m(f) : B\Sigma_m \to E^m_{h\Sigma_m} \xrightarrow{\mu} E, \in E^0(B\Sigma_m)$$

is called the $m$th total power operation. The composite

$$E^0 \xrightarrow{P^m} E^0(B\Sigma_m) \to E^0$$
$$x \mapsto P^m(x) \mapsto x^m$$

# Power operations

$P^m$ is multiplicative but not additive. $(x+y)^m = x^m + y^m + \cdots$

$$\Psi^m : E^0 \xrightarrow{P^m} E^0 B\Sigma_m \to E^0 B\Sigma_m / I$$

called the additive total power operation, which is a ring map.

**Theorem (Ando, Hopkins, Strickland, 04, [AHS04])**

*The ring $E^0 B\Sigma_m / I = 0$ for $m \neq p^k$. When $m = p^k$, the ring $A_k := E^0 B\Sigma_{p^k} / I$ classifies all degree $p^k$ isogeneis starting from $F_E$ over $E^0$.*

# Calculations

- $n = 1$ case: $E_1 = KU_p^\wedge$, $\Psi^p(x) = (1+x)^p - 1$
- $n = 2$ case: Using elliptic curves
  - when $p = 2$, Rezk [Rez08] computes $\psi^2 : (E_2)^0 \to (E_2)^0 B\Sigma_2/I$ as
    $$\mathbb{Z}_2[\![a]\!] \to \mathbb{Z}_2[\![a]\!][d]/(d^3 - ad - 2),\ a \mapsto a^2 + 3d - ad^2.$$
  - when $p = 3$, Yifei [Zhu14] computes $\Psi^3 : (E_2)^0 \to (E_2)^0 B\Sigma_3/I$ as
    $$\Psi^3 : E_2^0 \to E_2^0[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3)$$
    where $E_2^0 = \mathbb{Z}_9[\![h]\!]$, with
    $$\begin{aligned} h \mapsto & h^3 + (\alpha^3 - 6\alpha - 27)h^2 + 3(-6\alpha^3 + \alpha^2 + 36\alpha + 67)h \\ & + 57\alpha^3 - 27\alpha^2 - 334\alpha - 342. \end{aligned}$$

# Calculations

- For general $p$, Yifei [Zhu19] also obtained the explicit expression of $E_2^0 B\Sigma_p/I$ and $\Psi^p$, by translating the subgroup parameter $\alpha$ into a modular forms.
  $E_2^0 B\Sigma_p/I = E_2^0[\alpha]/w(h,\alpha)$ with

$$w(h,\alpha) = (\alpha - p)(\alpha + (-1)^p)^p - (h - p^2 + (-1)^p)\alpha.$$

• $n \geq 3$ case: Unknown. Lack of accessible algebraic models providing formal groups of height greater than 2.

# An Calculation of the power operation over $L_{K(1)}E_2$

$F = L_{K(1)}E_2$ is the Bousfield localization of $E_2$ respect to $K(1)$, with

$$\pi_* L_{K(1)}E_2 = W(k)((h))_p^\wedge[u^\pm]$$

Let $\alpha^*$ be the unique solution of $w(h, \alpha)$ in $F^0$, then

## Theorem

*The total power operation $\psi_F^p$ on $F^0$ is determined by*

$$\psi_F^p(h) = \alpha^* + \sum_{i=0}^{p}(\alpha^*)^i \sum_{\tau=1}^{p} w_{\tau+1} d_{i,\tau}, \qquad (1)$$

$$\alpha^* = (-1)^{p+1} p \cdot h^{-1} + \left(1 + (-1)^{p+1}\frac{p(p-1)}{2}\right) p^3 \cdot h^{-3} + \cdots$$

*In particular, $\psi_F^p$ satisfies the Frobenius congruence, i.e.* $\psi_F^p(h) \equiv h^p \mod p$.

# Modular interpretations of $L_{K(n-1)}E_n$

**Theorem**

*The ring $L_{K(n-1)}E_n^0(B\Sigma_{p^r})/I$ classifies augmented deformations of formal group of height $n-1$ together with a rank $p^r$ subgroup.*

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**Theorem**

*All spectra coming from augmented deformations of height $n-1$ over a field $k$ are homotopy equivalent.*

# Future Works

Though these spectra are homotopy equivalent, these equivalent usually do not preserve the $E_\infty$ ring structure over them.

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It is known that $E_n$ carries an essentially unique $E_\infty$ structure, while the $E_\infty$ structures over $L_{K(n-1)}E_n$ are not unique, at least for $n = 2$. [Van21]

# Future Works

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It is known that $E_n$ carries an essentially unique $E_\infty$ structure, while the $E_\infty$ structures over $L_{K(n-1)}E_n$ are not unique, at least for $n = 2$. [Van21]

Question: How many kinds of $E_\infty$ structures can $L_{K(n-1)}E_n$ carry? What does the moduli space/stack of this problem look like?

The answer may lie in the $\infty-$category world...

# Thank You!

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